Extremal Networks and Connectivity

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Certificate of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying subject to the provisions of the Copyright Act 1968.

(Signed) ____

Kim Marshall

Dedication

To the memory of my paternal grandmother, Margaret Isabel Marshall,
(16th August 1910 - 6th February 2005).
Grandma removed my father from school at the age of 14, due to the conviction that formal education was a waste of time. Let us hope that she is wrong on that point.

To the memory of my maternal grandfather, John Francis Lynch,
(13th April 1918 - 29th October, 1993).
Pop beamed with pride at the graduation of his first grandchild, my brother. His departure prompted my return to my home, my country and my studies.

Acknowledgements

I have not the words to thank my principle supervisor Professor Mirka Miller. There were a number of times throughout my PhD candidature that I thought I could not complete my studies, due to financial concerns, geographical obstacles and, more recently, family commitments. Mirka was always obstinate that I would finish the journey that I had begun and seemingly insurmountable obstacles disappeared. My sincere thanks and gratitude go to Mirka for her persistence, patience, faith, flexibility, encouragement, advice, friendship and support.

During my PhD candidature I was very privileged in that I had three different supervisors, namely, my principle supervisor, Professor Mirka Miller, my co-supervisor Dr Joe Ryan and my external co-supervisor Professor Camino Balbuena. I would like to thank them all for sharing their time and expertise with me and for their support and counsel during my candidature. I thoroughly enjoyed working with my supervisors and sincerely hope that the enjoyment was mutual. I particularly enjoyed Dr Joe Ryan's ability to lift morale with his remarkable sense of humour and incredible ability to recite poetry and prose. I will always admire my external co-supervisor Professor Camino Balbuena for her passion, determination and warmth. Although this journey is at an end, I will treasure the memories I have and hope that our paths cross frequently in the future.

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I would like to thank Geoffrey Exoo firstly, for making his research publicly available on his website. Secondly, for his kindness in providing me with adjacency lists of some graphs that were not available from his website. Thirdly, for generously allowing me unrestricted use of a program that he had written which turned out to be useful for verifying a number of my results and, finally, for some interesting discussions.

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My sincere gratitude goes to the people who cared for my children without them I would have been unable to complete my studies: Günter and Lieselotte Furche, who took on the role of caring for my children out of a conviction that I need to be emancipated and fortuitous my childrens' German language skills have improved beyond my own capabilities; Jane with the big truck and the million dollar duck, who, incidentally, introduced the children to Shakespeare's Macbeth “Bubble bubble, toil and trouble”, my father and my brothers' families.

Last but not least, I would like to thank my husband and children. Firstly, for providing a great many interesting and pleasant diversions from my studies. Secondly, for enticing me to complete this journey with the promise of resuming the journey that we have begun together.

List of Publications

Publications Arising from This Thesis

1. C. Balbuena, J. Tang, K. Marshall, Y. Lin, Superconnectivity of regular graphs with small diameter, *Discrete Applied Mathematics* **157** (2009), pp. 1349- 1353.
2. C. Balbuena, K. Marshall, L.P. Montejano, On the connectivity and superconnected graphs with small diameter, *Discrete Applied Mathematics* **158** (2010), pp. 397-403.
3. K. Marshall, M. Miller and J. Ryan, Extremal Graphs without Cycles of length 8 or less *Eurocomb 2011*
4. C. Delorme, E. Flandrin, Y. Lin, K. Marshall, M. Miller and J. Ryan, Extremal graphs with small girth, *In preparation.*
5. K. Marshall, M. Miller and J. Ryan, On maximum size of graphs with girth greater than 8, *In preparation.*

Further Publications Produced During my Candidature

6. K. Marshall, J. Ryan, Mode and antimode graphs, *Proceedings of the Sixteenth Australasian Workshop on Combinatorial Algorithms*, (2005), pp. 231- 238.
7. K. Marshall, J. Ryan, On antimode graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing*, (May 2008), pp. 51-60.
8. K. Marshall, J. Ryan, The eccentric nature of generalised Petersen graphs, *In preparation.*

Abstract

In 1736, in the town of Königsberg it was asked, "Is it possible to walk across the seven bridges that span the river Pregel, which divide the town of Königsberg into four land masses, without having to cross any bridge more than once?" Euler observed that it was unnecessary to consider the size of the land masses, the length of the bridges or the route taken to traverse the bridges in order to answer this question. He showed that the problem could be abstracted by considering only the *topology of the network*, where a network is a group or system of interconnected things or nodes: in this case the four land masses; and a topology is the way in which constituent parts are interrelated or linked: in this case, whether there is a bridge between any two chosen landmasses or not. The Königsberg bridge problem was the first problem in recorded history to be formulated in graph theoretic terms, that is, as the topology of a network. Euler's solution to this problem is considered to be the first theorem of graph theory.

Today, 275 years later, graph theory is a vibrant field of research with remarkably diverse applications, including: molecular chemistry; developing vaccination strategies to prevent the spread of viruses through human populations and computer networks; modelling complex ecological systems; analysis of social networks; and the design of VLSI (very large scale integrated circuits) of multiprocessors. In this thesis we consider questions in two separate but related research areas in the field of graph theory, namely, extremal graph theory and connectivity.

Extremal graph theory is the study of graphs that are extremal, that is, maximal or minimal, under some given constraints. In this thesis we focus on the problem of finding the maximum number of pair-wise connections between the nodes in a network, given the number of nodes and the length of the shortest cycle in the network. A graph that attains this bound is called an extremal graph. Our interest in extremal graphs arose from the problem of determining the structure of the most efficient and reliable networks. We provide constructions that produce infinite families of extremal graphs. We examine the relationship between extremal graphs and some other graphs that have been considered in the design of optimal networks. We develop an algorithm that we use to establish new and improved lower bounds on the size of some extremal graphs and determine the exact size of the extremal graphs for some particular parameters.

A graph is connected if there is a path, consisting of nodes and links, between any two nodes in the graph. The ability to send and receive email via the Internet is dependent upon the Internet being connected, that is, there is a path of computers and connections between the sender and receiver of the email. The connectivity of a network is the number of nodes or links that must be removed in order to partition the network into two or more components. High connectivity of a network corresponds to the properties of fault tolerance and resilience under attack. In this thesis we determine a number of sufficient conditions that ensure good connectivity of a network.

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Where's the orchestra?
After all, this is my big night on the town
My introduction...

Billy Joel

1 Introduction

In this chapter we provide some background to the field of graph theory and describe some applications that serve as a motivation for our research. In Sections 1.2 and 1.3, we introduce the two research areas of graph theory which are central to this thesis, namely, extremal graph theory and connectivity. In Section 1.4, we give a synopsis of the structure of the thesis including a summary of the main results.

1.1 Background

Graph theory is a research area in the disciplines of mathematics and computer science. A *graph* is a mathematical structure used to model pairwise relations between objects from a certain collection. The objects are called *vertices* and the relationships are called *edges*. When graphs are used to model computer networks, the vertices may be referred to as *nodes*, and the terms *links* or *connections* may be used interchangeably with edges. Likewise, the term *network* is often used in place of the term graph.

Although the first book written on graph theory "Theorie der endlichen und unendlichen Graphen" [83] did not appear until 1936, two-hundred years after Euler's solution to the Königsberg bridge problem, the applications of graph theory were already remarkably diverse; for example, in 1875, Cayley [38] wrote about the application of the graph theoretic problem of graph enumeration in particular, tree enumeration, to molecular chemistry. Equally remarkable is the number of famous graph theory problems that have no real world application.

Today, graph theory is a vibrant field of research with remarkably diverse applications. Graph theory has applications in medical, biological, epidemiology [39], conservation [29, 69], sociological [109] and technological research [72] as well as molecular chemistry [38]. Graphs are used to model and investigate the topologies of various networks. Examples include: social networks like Facebook [60] and Google + [79]; technical networks including VLSI (very large scale integrated circuits) of multiprocessors and the Internet; transport networks; distribution networks, such as blood vessels or postal delivery routes; and collaboration networks; for example, LinkedIn [96].

In this thesis we consider questions in two separate but related research areas in the field of graph theory, namely, extremal graph theory and connectivity. The questions that we consider have real world applications in the design and investigation of efficient, reliable, fault tolerant networks. In spite of that, we believe that these questions are worthy of investigation purely based on their mathematical merit.

The efficiency of a network can be measured in terms of the cost, or time needed to transmit a message. One measure of network efficiency is the *distance* (length of a shortest path) between all pairs of nodes in the network. Minimising the distance between vertex pairs reduces the time taken to transmit a message. One distance based parameter is the *diameter*, that is, the largest distance between any two vertices in the graph.

Another property of a network is reliability or fault tolerance. This property corresponds to being able to transmit data in a network even when some components fail. An example of fault tolerance is the ability to reroute an email when a mail server or satellite link is down. Fault tolerance of an air traffic network may be considered as the ability to reach your destination via an alternate airport or series of airports if one is closed due to volcanic ash, for example.

A network in which every node is linked to every other node has the best possible fault tolerance and efficiency but in the real world physical, geographical, political, logistical and economic constraints make it impossible, or at least highly impractical, to build such networks. However, it is desirable for a network to accommodate a large number of components while maintaining a low communication latency and good fault tolerance. In this thesis we consider a class of extremal graphs that fulfil these requirements. Furthermore, we determine a number of constraints that can be used to determine the fault tolerance of a graph in terms of connectivity.

1.2 Extremal Graph Theory

Extremal graph theory is the study of graphs that are extremal, that is, maximal or minimal, under some given constraints. Extremality can be taken with respect to different graph invariants, such as order, size, diameter, girth, connectivity, and maximum and minimum degree. In 1975, Erdős posed the problem of finding the maximum size of graphs that do not contain three-cycles or four-cycles. In this thesis, we examine a generalised version of this problem, namely, given parameters n and t, determine the maximum possible number of edges in a graph on n vertices that does not contain any cycles of length t or less as a subgraph. We use the term *extremal number* to indicate this value. Graphs having size equal to the extremal number are called *extremal graphs*.

Our interest in extremal graphs stems from the conviction that these graphs have the topology of a network that is both efficient and fault tolerant. It turns out that an extremal graph that does not contain any cycles of length t or less is necessarily of diameter less than t. This constraint on the diameter ensures small latency for message transmission. Evidence that the reliability of a network is enhanced when the smallest cycle in a graph is larger [98] prompted us to consider extremal graphs for larger values of t.

In this thesis we construct nine infinite families of graphs, which we prove to be extremal. Consequently, we establish two infinite series of previously unknown extremal numbers. Furthermore, we developed an algorithm which we use to find new and improved lower bounds on the extremal number $ex(n;t)$, for $t = 4, 5, \ldots, 11$ and $n \leq 200$. We also find the extremal number and some corresponding extremal graphs for some specific values of n and t.

1.3 Connectivity

The research area of connectivity is considered to have started, in 1927, with Menger's theorem (see Section 3.3). *Connectivity* is a fundamental property of graphs. A graph is *connected* if there is a path, consisting of nodes and links, between any two nodes in the graph. When considering the Internet, the property of being connected corresponds to being able to send email between any two people who are accessing the Internet.

Given a connected graph, the *connectivity* of the graph is the equal to cardinality of the minimum set of nodes (vertex cut) that must be destroyed or removed in order that the graph is no longer connected. In general, a graph with high connectivity is more robust against attacks or faults than a graph with low connectivity. A graph is *superconnected* if all minimum vertex cuts are trivial, that is, result in one of the components being an isolated node.

Connectivity has obvious applications to large scale complex networks in computer science. Other applications can be found in epidemiology, where removal of vertices in a contact network might correspond to vaccination of individuals against a disease [39]. The rezoning of Australia's Great Barrier Reef for marine conservation was determined using a graph-theoretic approach taking into account the connectivity of different populations by means of ocean currents [29,69].

In this thesis we determine a number of structural properties of a graph that ensure good connectivity.

1.4 Structure of the Thesis

The remainder of the thesis is structured as follows.

Chapter 2: Definitions and Notation. In this chapter, we introduce the graph theoretic notation and terminology that will be used throughout the thesis.

Chapter 3: Literature Review. In this chapter, we provide an historical overview of some results in the areas of extremal graph theory and connectivity. The purpose of this chapter is not to provide an exhaustive list of known results but rather to furnish the background required for the presentation of our results in Chapters 4, 5 and 6.

Chapter 4: Extremal Graphs. Our new results in the area of extremal graphs are presented in this chapter. More precisely, we describe our "Growing and Pruning" algorithm which we use to create new lower bounds on $ex(n;t)$, for $t = 4, 5, \ldots, 11$ and $n \leq 200$. Note that, for $t = 4$ and $t = 6$, many of these new lower bounds are improvements on the best known lower bounds on $ex(n;t)$ that were recently published by Abajo and Diánez [2] and Abajo, Balbuena and Diánez [1]. Moreover, we show that a number of graphs when subdivided form infinite families of extremal graphs, namely, the complete graphs K_2, K_3 and K_4, the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,4}$, the Petersen graph, the Heawood graph and the Tutte-Coxeter cage. Additionally, we establish the exact values of the previously unknown extremal numbers: $ex(n;6)$, for $n = 30, 31, 32$; $ex(n;8)$, for $n = 23, 24, 25, 26$; $ex(n;9)$, for $n = 26, 27, 28, 29$; and $ex(127;11)$. Some of these results have been accepted for publication [103]. The remaining material is being prepared for submission.

Chapter 5: Connectivity. In this chapter, we improve upon a result by Balbuena and Marcote [18] by showing that any graph G is 2-connected if diameter $D \leq g - 1$ for even girth g, and for odd girth g and maximum degree $\Delta \leq 2\delta - 1$, where δ is the minimum degree. Furthermore, we extend the results of Balbuena, Carmona, Fàbrega and Fiol [11], by proving that any graph G of diameter $D \leq g - 2$ is 5-connected for even girth g and $\Delta \leq 2\delta - 1$. This material has been published in [19].

Chapter 6: Superconnectivity. In this chapter, we improve known results by Fàbrega and Fiol [58] on the superconnectivity of a graph. We prove that an r-regular graph with odd girth g, $r \geq 3$ and diameter $D \leq g - 2$ is super-κ. Furthermore, we extend these results by showing that non regular graphs with odd girth g and diameter $D \leq g - 2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2 - 1$, are super-$\kappa$. This work has been published in [19] and [20].

Chapter 7: Conclusion. In this chapter, we summarise the main results of the thesis and present some open problems for future research.

The main contributions of this thesis are contained in Chapters 4, 5 and 6. All original results are indicated by the symbol $\diamondsuit$ and ends of proofs are marked by the symbol $\blacksquare$.

In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generations adds a new story to the old structure.

Hermann Hankel

2

Definitions and Notation

In this chapter we establish the terminology, definitions and notation that will be used throughout this thesis. Terminology and notation specific only to a particular chapter will be defined therein. In general we follow the notation used by Chartrand, Lesniak and Zhang in [41].

2.1 Basic Concepts

A *graph* is an ordered pair of sets $(V, E) = (V(G), E(G))$, where $V = V(G)$ is nonempty, and $E = E(G)$ is a set of unordered pairs of elements of $V(G)$. The elements of $V(G)$ are called the *vertices* of G and the elements of $E(G)$ are called the *edges* of G. When graphs are used to model computer networks, the vertices may be referred to as *nodes* and the terms *links* or *connections* may be used interchangeably with edges.

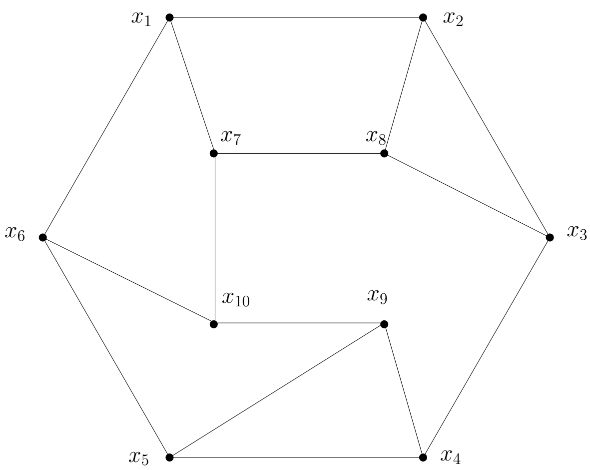


Figure 2.1: A graph G.

The *order* of a graph G is the number of vertices in G, denoted $n = n(G) = |V(G)|$, and the number of edges in G is referred to as the *size* of G, denoted $m = m(G) = |E(G)|$. The graph G drawn in Figure 2.1 has vertex set $V = \{x_1, x_2, \ldots, x_{10}\}$, edge set $E = \{x_1x_2, x_1x_6, x_1x_7, x_2x_3, x_2x_8, x_3x_4, x_3x_8, x_4x_5,\ x_4x_9, x_5x_6, x_5x_9, x_6x_{10}, x_7x_8,\ x_7x_{10},\ x_9x_{10}\}$, order $n = |V(G)| = 10$, and size $m = |E(G)| = 15$.

If u and v are vertices of a graph G, and there exists an edge $e = uv$, then we say that the edge e *joins* the vertices u and v, alternatively, we say that u is *adjacent* to v. The vertices u and v are said to be *incident* with the edge e and the vertices u and v are called the *endpoints* of e.

If two vertices are incident with more than one edge then we call these edges *multiple edges* or *multiedges*, and the graph is called a *multigraph*. An edge that is incident with only one vertex twice is called a *loop* and a graph that may contain loops is referred to as a *pseudograph*. Graphs without loops and multiple edges are called *simple* graphs. Multigraphs can be used to model network properties, for example, redundancy. However, in this thesis, we restrict our research to simple, finite graphs, that is, graphs on a finite number of vertices, without loops and/or multiple edges.

We say that u is a *neighbour* of v and the set of all neighbours of v is called the *neighbourhood of* v, denoted $N_1(v)$ or $N(v)$, for example, the vertices x_1 and x_3 from the graph G shown in Figure 2.1, have the neighbourhoods $N(x_1) = \{x_2, x_6, x_7\}$ and $N(x_3) = \{x_2, x_4, x_8\}$, respectively.

The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between u and v. The *eccentricity* of a vertex, denoted $e(v)$, in a graph G, is the distance from v to a vertex in G that is furthest from v, that is, $e(v) = max\{d(v, u)|u \in V(G)\}$. The *diameter* of a graph G, written $D = diam(G) = D(G)$, is the maximum eccentricity of all the vertices of G, that is, $diam(G) = max\{e(v)|v \in V(G)\}$. The *radius* of G, denoted $rad(G)$, is the minimum eccentricity of any vertex in G, that is, $rad(G) = min\{e(v)|v \in V(G)\}$.

For $S \subset V$, $d(v, S) = d_G(v, S) = \min\{d(v, s) :\ s \in S\}$ denotes the *distance* between a vertex v and a set S. For every $v \in V$ and every positive integer $r \geq 0$, $N_r(v) = \{u \in V : d(u, v) = r\}$ denotes the *neighbourhood of* v *at distance* r. Similarly, for $S \subset V$, the neighbourhood of S at distance r is denoted $N_r(S) = \{v \in V : d(v, S) = r\}$. Observe that $N_0(S) = S$. When $r = 1$ we write $N(v)$ and $N(S)$, instead of $N_1(v)$ and $N_1(S)$.

The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident with v; it is the cardinality of the neighbourhood of v, that is, $deg(v) = |N(v)|$. An *isolated vertex* is a vertex that has no neighbours and a vertex having only one neighbour is called a *pendant vertex*. The *maximum degree* of G, denoted $\Delta = \Delta(G)$, is the maximum degree over all the vertices of G. Similarly, the *minimum degree* of G, denoted $\delta = \delta(G)$, is the minimum degree over all the vertices of G. If all vertices of G have the same degree r, then G is said to be *regular* of degree r or *r-regular*. A 3-regular graph is said to be *cubic* or *trivalent*. The graph G in Figure 2.1 is a trivalent graph. The *degree sequence* of a graph G, denoted $\mathcal{D} = \mathcal{D}(G)$, is the non-increasing sequence of its vertex degrees, that is, $\mathcal{D} = (deg(v_1), deg(v_2), \ldots, deg(v_n))$. Since $0 \leq deg(v) \leq n-1$ if there are n vertices in the largest connected component then, by pigeonhole principle, at least one degree occurs more than once. Let $d_1, d_2, \ldots, d_t$ be the vertex degrees in G, in descending order, and suppose degree d_i occurs x_i times in G. Then we use the more concise notation $\mathcal{D} = (d_1^{x_1}, d_2^{x_2}, \ldots, d_t^{x_t})$. The degree sequence for the graph G in Figure 2.1 is $\mathcal{D} = (3^{10})$ and the graph G in Figure 2.2 has degree sequence $\mathcal{D} = (3^4, 2^1, 1^2)$.

A graph S is a *subgraph* of G if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$. If $V(S) = V(G)$ then S is a *spanning subgraph* of G. If $V(S) \subsetneq V(G)$, then S is a called a *proper subgraph* of G. If $V(S) \subset V(G)$ then $G[S]$ denotes the *subgraph induced* by the vertex set $V(S)$ of $V(G)$. The graph in Figure 2.2 is the subgraph induced by the vertex set $V(S) = \{x_3, x_4, x_5, x_6, x_7, x_9, x_{10}\}$ of the graph G in Figure 2.1. The degree of a vertex v restricted to the induced subgraph S of G is denoted by $deg_S(v) = |N(v) \cap V(S)|$, for example, the vertex x_3 in the graph G of Figure 2.1 has $deg(x_3) = 3$, and $deg_S(x_3) = 1$ in the subgraph shown in Figure 2.2.

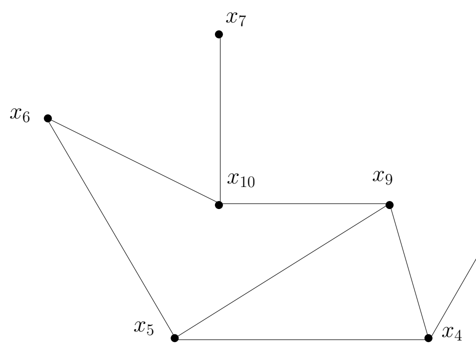


Figure 2.2: A subgraph S of the graph G in Figure 2.1.

The *edge-degree* of an edge uv is defined as $deg(uv) = deg(u) + deg(v) - 2$. The *minimum edge-degree* of G, denoted by $\xi = \xi(G)$, is defined as $\xi(G) = \min\{deg(u) + deg(v) - 2 : uv \in E(G)\}$.

A *walk* $W = v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots v_{n-1}, v_{n-1}v_n, v_n$ in a graph is an alternating sequence of vertices and edges. W is also referred to as a v_0-v_n *walk* of length n since it contains n edges. A walk of length n in which no edges are repeated is called a *trail*. A walk in which no vertices are repeated is called a *path*, denoted P_n. We call v_0 and v_n the *end vertices* of the path, and we say the vertices v_0 and v_n are *connected by the path* P_n. For convenience we may refer to a path by its end vertices, and use the notation v_0v_n*path* to refer to a path between v_0 and v_n. The graph in Figure 2.1 contains the walk of length 5, $W = x_1, x_1x_2, x_2, x_2x_8, x_8, x_8x_7, x_7, x_7x_1, x_1, x_1x_2, x_2$, the trail $T = x_1, x_1x_2, x_2, x_2x_8, x_8, x_8x_7, x_7, x_7x_1, x_1$ and the path $P_4 = x_1, x_2, x_8, x_7$.

A *Hamiltonian path*, also called a *Hamilton path*, is a path between two vertices of a graph G that visits each vertex of G exactly once. A Hamiltonian path that is also a cycle is called a *Hamiltonian circuit* or *Hamiltonian cycle*. A graph possessing a Hamiltonian circuit is said to be a *Hamiltonian graph*.

2.2 Families of Graphs

A *cycle* of length n, written C_n, is a graph with n vertices $v_1, v_2, ..., v_n$ and n edges $v_1v_2, v_2v_3, ..., v_nv_1$. The *length of a cycle* is equal to the number of vertices or, equivalently, edges in the cycle. A cycle of length three is frequently referred to as a *triangle*. The *girth* of a graph G, denoted by $g = g(G)$, is the length of a shortest cycle in G; the length of a longest cycle in G is its *circumference*. A cycle having length equal to the girth is called a *girth cycle*. The girth of any complete graph K_n is 3 and the circumference is n. If G does not contain a cycle then G is said to be *acyclic* and the girth is considered to be 0 and the circumference ∞.

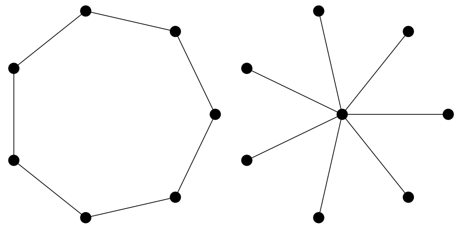


Figure 2.3: The cycle C_7 and the complete bipartite graph $K_{1,7}$.

An acyclic connected graph is called a *tree*, T. Vertices in the tree having degree 1 are called *leaves*. A vertex of T may be distinguished by being called the *root* in which case the *height* of the tree is defined as the maximum distance between the root and any leaf of the tree. Every connected graph G has a *spanning tree*, that is, a subgraph T such that $V(T) = V(G)$. The number $m - (n-1) = m - n + 1$ is referred to as the *cycle rank* of G.

The *complete graph* K_n is the graph on n vertices, where every vertex is adjacent to every other vertex, for example, Figure 2.4 contains drawings of the complete graphs K_7 and K_8. The complete graph K_n is regular of degree $n-1$, that is, $\mathcal{D} = ((n-1)^n)$ has $|E(K_n)| = \binom{n}{2} = n(n-1)/2$ edges. A proper subgraph which is a complete graph is called a *clique*.

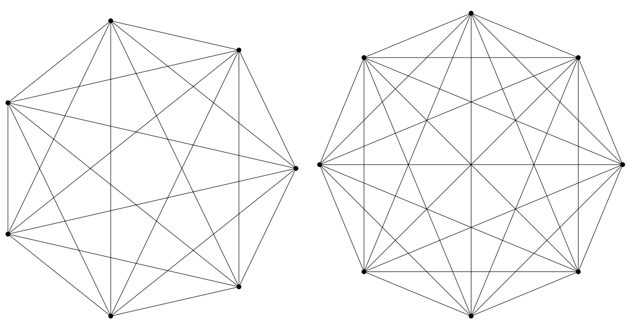


Figure 2.4: The complete graphs K_7 and K_8.

Another often studied class of graphs are the *bipartite graphs*. A graph $G = (V(G), E(G))$ is *bipartite* if the vertex set $V(G)$ can be partitioned into two subsets of vertices, $V(U)$ and $V(W)$, such that every edge $uw \in E(G)$ has $u \in V(U)$ and $w \in V(W)$. The complete bipartite graph, denoted $K_{m,n}$, is the graph on the union of the two disjoint sets of vertices $V(M)$ and $V(N)$, where $m = |V(M)|$ and $n = |V(N)|$, with every vertex in M adjacent to every vertex in N, and no two vertices in the same set adjacent to each other. The complete bipartite graph $K_{1,n-1}$ is also known as the *star on n vertices*, denoted S_n. The complete bipartite graph $K_{1,7}$ or S_8 star is shown in Figure 2.3. Figure 2.5, contains drawings of the complete bipartite graph $K_{4,5}$ and two different spanning trees of $K_{4,5}$.

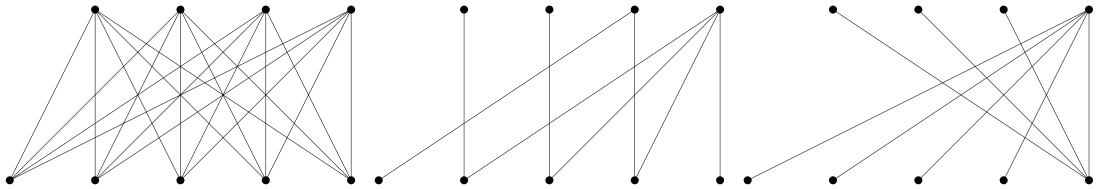


Figure 2.5: The complete bipartite graph $K_{4,5}$ and two different spanning trees of $K_{4,5}$.

The definition of complete bipartite graphs can naturally be extended to encompass *complete k-partite graphs*. A complete k-partite graph, denoted $K_{n_1,n_2,n_3,\ldots,n_k}$ is a graph of order $n \geq 3$, where the vertices are partitioned into $k \leq n$ sets of vertices. Edges exist between every pair of vertices that are in different partite sets and there are no edges between vertices in the same partite set. The complete k-partite graph with each vertex set having either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices is known as the *Turán graph*, denoted, $T_{n,k}$.

Another interesting class of graphs are the (k, g)-*graphs* which are k-regular graphs of girth g. In 1947, Tutte studied $(3, g)$-graphs and introduced the term "cage" to describe a $(3, g)$-graph with minimal order. The (3,5)-cage is the Petersen graph which is shown in Figure 2.8. The (3,6), (3,7) and (3,8)-cages, shown in Figure 2.6, are also known, respectively, as the Heawood graph, McGee graph and Tutte-Coxeter graph. The study of cages has since been generalised to include regular graphs with $k > 3$. More formally, an (k, g)-*graph* with minimum order for particular values of k and g is called a (k, g)-*cage*. The (7,5)-cage is also known as the Hoffman-Singleton graph.

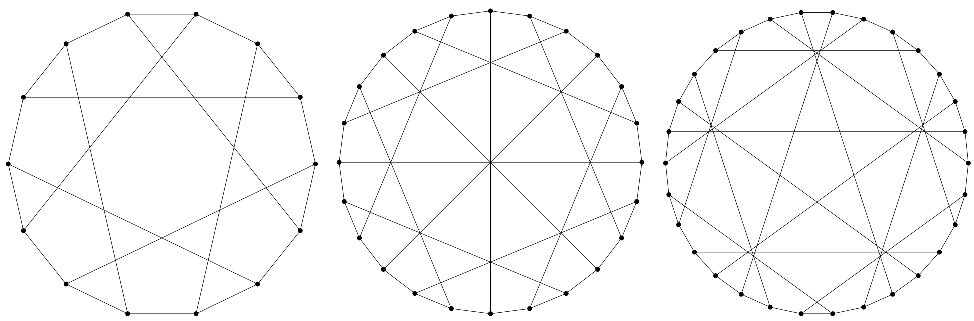


Figure 2.6: The Heawood graph, McGee graph and Tutte-Coxeter graph.

A *finite projective plane* consists of a finite set of *lines* and a finite set of *points*, and a relation between points and lines called an *incidence relation*, having the following properties:

- Given any two distinct points, there is exactly one line incident with the two points.
- Given any two distinct lines, there is exactly one point incident with the two lines.
- There are four points such that no line is incident with more than two of these points.

A finite projective plane has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, each line contains $n + 1$ points and every point has $n+1$ lines passing through it, where n is an integer called the *order* of the projective plane.

Let P be a finite projective plane, we define a *polarity of* P, denoted π, to be a one-to-one mapping of points onto lines such that $q \in \pi(p)$ whenever $p \in \pi(q)$. A *polarity graph* $G(P, \pi)$, defined by Erdős and Rényi [49], is a graph whose vertex set is the set of points of P and whose edge set is $\{pq : p \in \pi(q), p \neq q\}$. A polarity graph of a finite projective plane has diameter 2, no three or four cycles and order $n^2 + n + 1$, where n is the order of the projective plane.

The term *generalised polygon*, also known as a *generalised D-gon*, was introduced by Tits [120] in 1974. In order to define generalised polygons we first introduce the terms "incidence structure" and "incidence graph". An *incidence structure* is an ordered triple $\mathcal{I}= (P, L, I)$, where $P \neq \emptyset$ is a set of *points*, $L \neq \emptyset$ is a set of *lines*, and $I \subseteq P \times L$ is the point-line relation, called the *incidence relation*. Given an incidence structure $\mathcal{I}$, if each line is incident with exactly $s + 1$ points, and each point is incident with exactly $t + 1$ lines, we say that $\mathcal{I}$ has order (s, t). If $s = t$ then $\mathcal{I}$ is said to have order s. The *incidence graph* of $\mathcal{I}$, denoted $G(\mathcal{I})$, is the graph with vertex set $V(G(\mathcal{I})) = P \cup L$ with edge set $E(G(\mathcal{I})) = (p, l) \in I$, where $p \in P$ and $l \in L$. A *generalised polygon* is an incidence structure $\mathcal{I}= (P, L, I)$, such that,

- $\mathcal{I}$ has order (s, t), where $s \geq 1$ and $t \geq 1$
- Any two distinct lines intersect in at most one point
- There is at most one line through any two distinct points
- The incidence graph $G(\mathcal{I})$ is a bipartite graph of diameter D and girth $2D$

The incidence graph of a generalised digon ($D = 2$) is the complete bipartite graph $K_{s+1,t+1}$. For $D \geq 3$ and $s = t = 1$ we obtain the ordinary two-dimensional polygons with D sides. Feit and Higman [61] established the non-existence of certain generalised polygons, asserting that for $s > 1$ and $t > 1$ generalised polygons exist only when $D = 2, 3, 4, 6$ or 8.

A generalised triangle of order $s > 1$, is a projective plane of order s. To date projective planes of order s are known to exist only when s is a prime power. Figure 2.7 contains a generalised triangle or projective plane and the incidence graph of the generalised triangle, which in this case is the Heawood graph or (3,6)-cage.

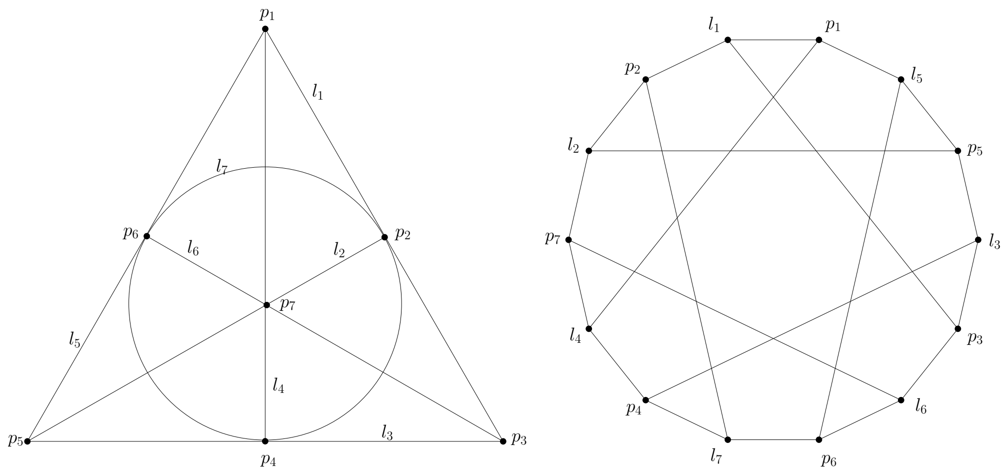


Figure 2.7: A generalised triangle, $\mathcal{I}$, and the incidence graph $G(\mathcal{I})$.

2.3 Operations on Graphs

The *union* of G and G', denoted by $G \cup G'$, is the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. The *intersection* of G and G', denoted by $G \cap G'$, is the graph with vertex set $V(G) \cap V(G')$ and edge set $E(G) \cap E(G')$. The *difference* between G and G', denoted by $G - G'$, is the graph with vertex set $V(G) - V(G')$ and edge set formed by all the edges of G with both end vertices in $V(G) - V(G')$.

Removing a vertex, or *vertex removal*, is the operation that deletes a vertex and all its incident edges from the graph. *Removing an edge*, or *edge removal*, is the operation that deletes an edge from the graph. *Edge contraction* is the operation that removes an edge $e = uv$ and then combines the two vertices u and v into a single vertex that is adjacent to all the former neighbours of u and v. A *minor* of a graph G is a graph that can be obtained by applying to G the operations of edge deletion, edge contraction or vertex deletion.

Given a graph G, *subdividing an edge $uv \in E(G)$ by i* results in a graph G' with vertex set $V(G') = V(G) \cup \{x_1, x_2, \ldots, x_i\}$ and edge set $E(G') = \{E(G) - \{uv\}\} \cup \{\{ux_1\}, \{x_1x_2\}, \ldots, \{x_iv\}\}$. Given a graph G of order n, size m, and girth g, *subdividing the graph G by i* is the operation of subdividing every edge in G. The resulting graph, denoted s_iG, has order $n + mi$, size $m(1 + i)$ and girth $g(1 + i)$, for example, the subdivided Petersen graph is shown in Figure 2.8.

The *complement of a graph G*, denoted $\overline{G}$, is the graph with the same vertex set as G, whose edge set consists of the edges that are not present in G, for example, the complement of the complete graph K_7 is a graph having 7 vertices and no edges, the complement of the complete bipartite graph $K_{3,4}$ is the disjoint union of K_3 and K_4 and the complement of the complete multi-partite graph $K_{5,5,6}$ consists of the complete graphs K_5, K_5 and K_6.

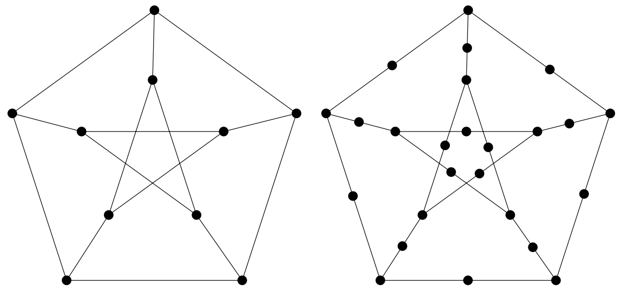


Figure 2.8: The Petersen graph P and the subdivided Petersen graph s_1P.

Two graphs G_1 and G_2 having the same order n are said to be *isomorphic* or *identical under isomorphism* if there is a one-to-one mapping f of the vertices $V(G_1)$ to the vertices $V(G_2)$ such that v_1 and u_1 are adjacent in G_1 if and only if the vertices $f(v_2)$ and $f(u_2)$ are adjacent in G_2.

2.4 Extremal Graph Theory

Extremal graph theory is the study of graphs that are extremal, that is, maximal or minimal, under some given constraints. Extremality can be taken with respect to different graph invariants, such as order, size, diameter, girth, connectivity, maximum and minimum degree. More abstractly, it is the study of how global properties of a graph influence local substructures of the graph.

In 1941 Turán, [46] asked: "How many edges must a graph contain that it should certainly have subgraphs of a prescribed structure?". Alternatively, what is the maximum size of a graph G having order n and the property that if $F \in \mathcal{F}$ then $F \not\subseteq G$. The answer to this question is called the *extremal number*, denoted $ex(n; \mathcal{F})$, and graphs with property $\mathcal{F}$ that have size equal to the extremal number are said to be *extremal graphs*, denoted $EX(n; \mathcal{F})$. Research concerned with this question is referred to as *extremal graph theory*.

In 1975, Erdős posed the problem of finding the maximum size of a graph on n vertices that does not contain three-cycles or four-cycles. In this thesis we examine a generalised version of this problem, namely, finding the maximum number of edges in a graph of order n that contains no cycle C_k, where $k \leq t$ and $t \geq 3$. We use the notation $ex(n; t) = ex(n; \{C_3, C_4, \ldots, C_t\})$ and the term *extremal number* to indicate this value. Graphs having size equal to the extremal number are called *extremal* $\{C_3, C_4, \ldots, C_t\}$*-free graphs* or just *extremal graphs* when the context is understood. For given values of n and t the set of extremal graphs is denoted $EX(n; t) = EX(n; \{C_3, C_4, \ldots, C_t\})$.

2.5 Connectivity

In this section we introduce definitions, notation and terminology that will be used to explain our results in connectivity.

A graph G is *connected* if there is a path between each pair of vertices in G, and is *disconnected* otherwise. The graph in Figure 2.9 is connected. Every disconnected graph can be split up into a number of maximal connected subgraphs, called *components*.

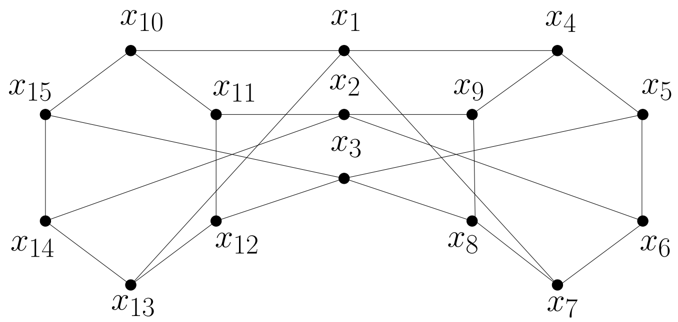


Figure 2.9: A connected graph.

In a connected graph, two or more uv-paths are *edge disjoint* if they have no edges in common, and *vertex disjoint* if they have no vertices in common apart from u and v. Certain vertices are said to *separate u from v* if the removal of these vertices destroys all paths between u and v. Similarly, certain edges *separate u from v* if the removal of these edges destroys all paths between u and v. The paths $x_{15}x_3x_5$ and $x_{15}x_{14}x_{13}x_{12}x_3x_8x_7x_6x_5$ in Figure 2.9 are edge disjoint but not vertex disjoint. The paths $x_{10}x_1x_4$ and $x_{10}x_{11}x_2x_9x_4$ in Figure 2.9 are both edge and vertex disjoint.

A vertex x is said to be a *cut vertex* of a graph G if removing x disconnects the graph. In this case there must be two vertices u and v such that x lies on every uv path in G. Similarly, an edge e in a connected graph G, whose removal disconnects the graph is called a *bridge*. Again, there must be two vertices u and v such that e lies on every uv path in G. A necessary and sufficient condition for an edge to be a bridge is that the edge lies on no cycle. Therefore, every edge of an acyclic connected graph is a bridge.

If G is a connected graph and $X \subset V(G)$ such that $G-X$ is not connected, then X is said to be a *vertex cut* or a *disconnecting set* of G. Analogously, if $F \subset E(G)$ and $G-F$ is not connected, then F is said to be an *edge cut* or an *edge disconnecting set*, for example, the graph in Figure 2.9 has a cut set $X = \{x_2, x_3, x_{10}, x_{13}\}$ and an edge disconnecting set $F = \{x_2x_6, x_3x_5, x_4x_5, x_6x_7\}$.

We say that G is *r-connected* if the deletion of at least r vertices of G is required to disconnect the graph. The *vertex connectivity*, denoted $\kappa = \kappa(G)$, or *edge connectivity*, written $\lambda = \lambda(G)$, of a connected graph G is the *smallest* number of vertices, respectively edges, whose removal disconnects G. More formally,

$$\kappa = min\{|X| : X \subseteq V(G) \text{ and } \omega(G - X) > 1\},$$

$$\lambda = min\{|F| : F \subseteq E(G) \text{ and } \omega(G - F) > 1\},$$

where $\omega(G - X)$ and $\omega(G - F)$ are the number of components of the graph obtained from G by removing the vertices of X, respectively edges of F. For a number of classes of graphs the vertex and edge connectivity has been defined or determined, for example: $\kappa(K_n) = n-1$ or ∞, $\lambda(K_n) = n - 1$, $\kappa(C_n) = 2$, $\lambda(C_n) = 2$, $\kappa(P_n) = 1$, $\lambda(P_n) = 1$, the vertex and edge connectivity of a tree is also 1. The graph in Figure 2.9 has vertex connectivity, $\kappa = 3$ and edge connectivity $\lambda = 3$.

There is an intrinsic relationship between the minimum degree, the edge connectivity and the vertex connectivity of a graph. Given a vertex $v \in G$, such that $deg(v) = \delta$, removing the set of all vertices that are neighbours of v, or alternatively the set of edges that are incident to v, necessarily disconnects the graph into at least two components one of which is the vertex v. This relationship between the vertex connectivity, edge connectivity and minimum degree, was first observed by Whitney [124] and can be stated $\kappa \leq \lambda \leq \delta$. The fact that the inequalities can be equalities is demonstrated by the cut set $X = \{x_1, x_5, x_9\}$ and the edge cut set $F = \{x_4x_1, x_4x_5, x_4x_9\}$ of the graph in Figure 2.9. The removal of these sets disconnects the graph into two components one of which is the isolated vertex x_4. A vertex cut (edge cut) whose removal results in two components one of which is an isolated vertex is called *trivial vertex cut* (*trivial edge cut*).

A graph G is said to have *maximum edge connectivity* if $\lambda = \delta$ and *maximum vertex connectivity* when $\kappa = \lambda = \delta$. A graph with maximum vertex connectivity is said to be *maximally connected.* The graph in Figure 2.9 is maximally connected.

Due to the fact that the minimum degree of a graph is less than or equal to the average degree, we can extend Whitney's [124] observation as follows, $\kappa \leq \lambda \leq \delta \leq 2m/n$, where n and m are, respectively, the order and the size of the graph, and $2m/n$ represents the average of the vertex degree. A graph for which these inequalities are equalities is said to have *optimal connectivity.* To show that a graph has optimal connectivity, it is sufficient to show that $\kappa = 2n/m$. By definition, for a graph to be optimally connected it must be degree regular.

Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.

Béla Bollobás [32]

3

Literature Review

In this chapter, we provide an historical overview of some important results in the research areas of extremal graph theory and connectivity. The purpose of this chapter is not to give an exhaustive list of results but rather to provide the background required to frame our new results which are presented in Chapters 4, 5 and 6.

In Section 3.1, we introduce two extremal graph theory problems concerned with the order of a graph. Firstly, determining the maximum possible order of a graph given constraints on the maximum degree and the diameter of the graph. Secondly, determining the minimum possible order of a degree regular graph with a prescribed girth. We also discuss some generalisations of these two problems.

In Section 3.2, we discuss a number of extremal graph theory problems, where the goal is to find the maximum possible number of edges in a graph that does not contain a certain subgraph. In particular, we introduce a problem which is central to the thesis, namely, determining the maximum possible size of a graph that does not contain any cycles of length t or less as a subgraph. This question is more thoroughly dealt with in Chapter 4, where we present some of our new results concerning extremal graphs.

In Section 3.3, we present historical results in the area of connectivity. In particular, a number of theorems that enable us to determine the minimum connectivity of a graph G using knowledge of the girth and diameter of G. We improve on some of these results in Chapters 5 and 6.

3.1 Graphs with Maximal/Minimal Order

In this section, we examine two well known extremal graph theory problems regarding the order of a graph. In Section 3.1.1, we begin with the degree/diameter problem, where the goal is to find the maximum possible number of vertices in a graph given constraints on the maximum degree and diameter. Furthermore, we illustrate a natural upper bound, known as the Moore bound, for this problem. Furthermore, we summarise current known results on the existence of graphs that attain the Moore bound and discuss some generalisations of the degree/diameter problem. We also include a generalisation of the degree/diameter problem, namely, the degree/diameter problem for bipartite graphs and summarise results in this area of research which will be relevant later in the thesis.

In Section 3.1.2, we state the degree/girth problem, where the goal is to find the minimum possible order of a degree regular graph with prescribed girth. We also discuss some variations of the degree/girth problem.

3.1.1 The Degree/Diameter Problem

Most of the material presented in this section is from the survey of the degree/diameter problem by Miller and Širáň [106]. The degree/diameter problem has also been considered for restricted families of graphs including vertex transitive, planar and bipartite graphs. For a more comprehensive review of the degree/diameter problem including: generalisations of the degree/diameter problem; construction techniques used to create large graphs; and a number of open problems for consideration, we recommend this survey.

The degree/diameter problem can be stated:

> *Degree/diameter problem:* Given natural numbers $\Delta \geq 2$ and $D \geq 1$, find the largest possible number $n_{\Delta,D}$ of vertices in a graph of maximum degree Δ and diameter $\leq D$.

The application of this problem to network design is obvious. The limitations on the degree correspond to physical limits on the number of connections that a component in the network can have, for example, the network may contain routers that have a limited number of ports. The limit on the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes, which is a measure of network efficiency.

A graph with maximum degree Δ and diameter D is called a (Δ, D)-*graph.* A natural upper bound on $n_{\Delta,D}$ can be determined by counting the maximum possible number of neighbours of a vertex in $r \in G$ at distance less than or equal to the diameter, that is, counting $|N_i(r)|$ for $0 \leq i \leq D$, as illustrated in Figure 3.1. This upper bound is known as the *Moore bound*, denoted $M_{\Delta,D}$ and a graph whose order is equal to the Moore bound is called a *Moore graph.*

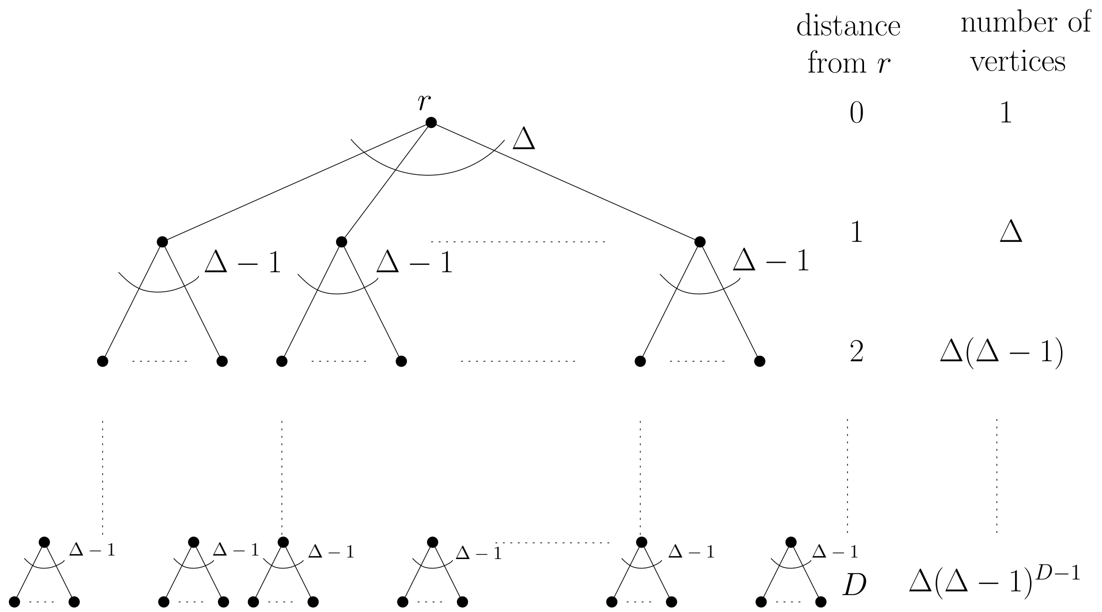


Figure 3.1: Illustration of Moore bound.

$$\begin{aligned} M_{\Delta,D} &= 1 + \Delta + \Delta(\Delta-1) + \ldots + \Delta(\Delta-1)^{D-1} \\ &= \begin{cases} 1 + \Delta\frac{(\Delta-1)^D - 1}{\Delta-2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{3.1}$$

Moore Graphs

The term Moore graph was introduced by Hoffman and Singleton [78], in 1960. In 1968, Singleton [115] proved that there is no irregular Moore graph. In other words, a graph that attains the Moore bound must be degree regular. Furthermore, by considering the illustration of the Moore bound in Figure 3.1 it is easy to determine that a graph that attains the Moore bound must have diameter D and odd girth $g = 2D + 1$.

The complete graphs $K_{\Delta+1}$ are Moore graphs, for diameter $D = 1$. The cycles of odd length C_{2D+1} are Moore graphs, for maximum degree $\Delta = 2$ and diameter $D \geq 2$. The existence of Moore graphs of diameter 2 and 3 was considered by Hoffman and Singleton [78], who proved that for $D = 2$ Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57, but not for any other degrees. Furthermore, they showed that for $D = 2$ and $\Delta = 3$ the Petersen graph is the only graph that obtains the Moore bound. Similarly, for $D = 2$ and $\Delta = 7$ the Hoffman-Singleton graph is the unique Moore graph. In 1973, Damerall [42], and independently, Bannai and Ito [22], used eigenvalue techniques to prove the non-existence of Moore graphs for $\Delta \geq 3$ and $D \geq 3$.

Due to the scarcity of Moore graphs, research on the degree/diameter problem is focused on finding large graphs that are in some way "close" to the Moore bound, that is, graphs of order $M_{\Delta,D} - \epsilon$. The parameter ϵ is called the *defect*. Such a graph is called a $(\Delta, D, -\epsilon)$-*graph* or *Moore graph of defect* ϵ.

Erdős, Fajtlowicz and Hoffman determined that the cycle C_4 is the only $(\Delta, 2, -1)$-graph, that is, the only Moore graph of defect 1, diameter 2, maximum degree Δ. Bannai and Ito [21] and, independently, Kurosawa and Tsujii [86] generalised this result to include all diameters and showed that, with the exception of the even cycles C_{2D}, Moore graphs of defect 1 do not exist.

Research activities related to the degree/diameter problem include the construction of graphs that improve the best current known lower bounds and providing non-existence proofs of graphs of order close to the established upper bounds. Finding solutions to the degree/diameter problem for particular values of Δ and D is known to be difficult. After over 50 years of research, the question of the existence of a Moore graph of diameter 2 and degree 57 remains unanswered. Even the question on monotonicity of $n_{\Delta,D}$ in Δ and/or D is still an open problem [106].

The Degree/Diameter Problem for Bipartite Graphs

In this section we introduce the degree/diameter problem for bipartite graphs and summarise current known results.

The degree/diameter problem for bipartite graphs can be stated:

> *Degree/diameter problem for bipartite graphs:* Given natural numbers $\Delta \geq 2$ and $D \geq 2$, find the largest possible number $n^b_{\Delta,D}$ of vertices in a bipartite graph of maximum degree Δ and diameter $\leq D$.

A natural upper bound for $n^b_{\Delta,D}$ can be determined by counting the number of neighbours of an edge $e \in G$ at distance less than or equal to the diameter, as illustrated in Figure 3.2. This bound is called the *Moore bipartite bound*, denoted, $M^b_{\Delta,D}$. A bipartite graph of maximum degree Δ, diameter D and order equal to the Moore bipartite bound, is called a *Moore bipartite graph*. Such a graph is necessarily regular of degree Δ and has even girth $2D$.

$$\begin{aligned} M^b_{\Delta,D} &= 2(1 + (\Delta-1) + \ldots + \Delta(\Delta-1)^{D-1}) \\ &= \begin{cases} 2\frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta > 2 \\ 2D & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{3.2}$$

For maximum degree $\Delta = 2$ and diameter $D \geq 2$, the Moore bipartite graphs are the even cycles on $2D$ vertices. For diameter $D = 2$ and each $\Delta \geq 3$, the Moore bipartite graphs of degree Δ are

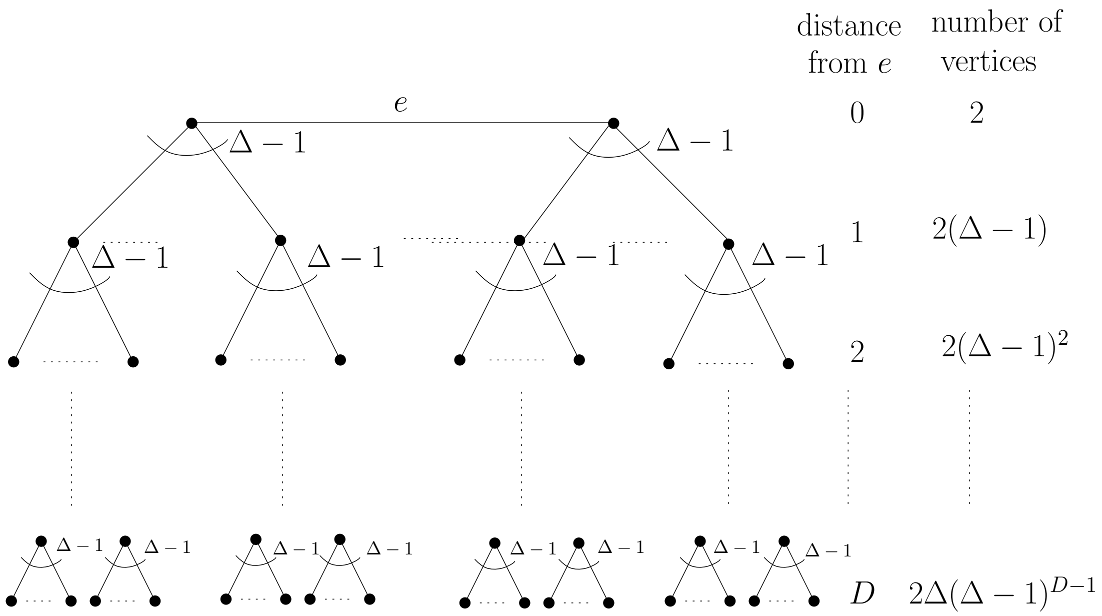


Figure 3.2: Illustration of bipartite Moore bound.

the complete bipartite graphs $K_{\Delta,\Delta}$. For maximum degree $\Delta \geq 3$, the rarity of Moore bipartite graphs was settled by Feit and Higman in their paper titled "The non-existence of certain generalized polygons" [59] which was published in 1964, and independently, by Singleton [114] in 1966. They proved that such graphs exist only if the diameter is 2,3,4 or 6. For $D = 3, 4, 6$ Moore bipartite graphs of degree Δ have been constructed only when $\Delta - 1$ is a prime power [24]. Furthermore, Singleton [114] proved that the existence of a Moore bipartite graph of diameter 3 is equivalent to the existence of a projective plane of order $\Delta - 1$.

Tables of the current best known upper bounds for the degree/diameter problem (including the degree/diameter problem for a number of restricted classes of graphs) are published on a section of the Combinatorics wiki website which is maintained by Loz, Pérez-Rosés and Pineda-Villavicencio [97].

3.1.2 The Degree/Girth Problem

In this section we draw extensively from the state-of-the-art dynamic cage survey by Exoo and Jajcay [56]. The survey includes all known cages, the best known upper bounds on the order of cages, construction techniques used to obtain these graphs and a number of open problems for consideration. We recommend this survey to any reader interested in a more comprehensive review of the degree/girth problem.

The degree/girth problem can be stated as:

> *Degree/girth problem:* Given natural numbers $k \geq 2$ and $g \geq 3$, find the least possible number $n(k,g)$ of vertices in a k-regular graph with girth g.

We use the notation (k,g)-graph to mean a k-regular graph with girth g. A (k,g)-graph having the least possible order is called a (k,g)-cage.

Cages

Recalling from Chapter 2, the term (k,g)-*cage* was originally introduced by Tutte [121] in 1947 to refer to a $(3,g)$-graph with the least possible number of vertices. The term "cage" was later generalised to include regular graphs with $k > 3$. The fact that such a (k,g)-cage exists for any given k and g was proven by Sachs [113] in 1963. In the same year, Erdős and Sachs [50] determined the following upper bound on the order of a (k,g)-cage,

$$n(k,g) \leq 4\sum_{t-1}^{g-2}(k-1)^t.$$

The current best known upper bound on the order of cages is due to Lazebnik, Ustimenko and Woldar [89] and can be stated as follows. Let $k \geq 2$ and $g \geq 5$ be integers and let q denote the smallest odd prime power for which $k \leq q$. Then,

$$n(k,g) \leq 2kq^{3/4g-a},$$

where $a = 4, 11/4, 7/2, 13/4$ for $g \equiv 0, 1, 2, 3 \bmod 4$, respectively.

Recalling from Section 3.1.1 the Moore bound, denoted $M_{\Delta,D}$, is the upper bound for the number of vertices in a graph with maximum degree Δ and diameter D and a graph that attains the Moore bound is necessarily degree regular with odd girth $2D+1$. Consequently, the Moore bound is also a lower bound on the number of vertices in a degree regular graph of odd girth. Therefore, any $(\Delta, 2D+1)$-graph with order equal to $M_{\Delta,D}$ is a cage. Regarding cages with even girth, Bond and Delorme [33] determined that the bipartite Moore bound $M^b_{\Delta,D}$ is also the lower bound on $n(k,g)$, for $g = 2D$. This relationship was later demonstrated by Biggs [26].

In order to express this lower bound in terms of girth rather than diameter we use the term *Moore bound for cages*, and the notation $M(k,g)$, to mean the Moore bound or the Moore bound for bipartite graphs, depending on the parity of g. Therefore,

$$M(k,g) = \begin{cases} 1 + k\sum_{i=0}^{(g-3)/2}(k-1)^i = \frac{k(k-1)^{(g-1)/2}-2}{k-2}, & \text{for odd } g; \\ 2\sum_{i=0}^{(g-2)/2}(k-1)^i = \frac{2(k-1)^{g/2}-2}{k-2}, & \text{for even } g. \end{cases}$$

In summary,

$$n_{\Delta,D} \le M_{\Delta,D} = M(k,g) \le n(k,g), \text{ for odd girth,}$$

and

$$n^e_{\Delta,D} \le M^b_{\Delta,D} = M(k,g) \le n(k,g), \text{ for even girth.}$$

We use the term *Moore cage* to refer to a (k,g)-graph with order equal to the lower bound $M(k,g)$. Drawing from the results for the degree/diameter and the degree/diameter problem for bipartite graphs discussed in Section 3.1.1, we make the following assertion.

There exists a Moore cage of degree k and girth g if and only if

- $k = 2$ and $g \ge 3$, cycles C_g
- $g = 3$ and $k \ge 2$, complete graphs K_{k+1}
- $g = 4$ and $k \ge 2$, complete bipartite graphs $K_{k,k}$
- $g = 5$ and
  - $k = 2$, the cycle C_5
  - $k = 3$, the Petersen graph
  - $k = 7$, the Hoffman-Singleton graph
  - and possibly $k = 57$
- $g = 6, 8$, or 12 and there exists a symmetric generalised n-gon of order $k-1$, where $k-1$ is a prime power

At this point we would like to discuss a number of somewhat conflicting uses of terminology that have been used to mean what we refer to as Moore cages. The above assertion appears in the dynamic cage survey by Exoo and Jajcay [56] with the exception that the authors refer to the graphs as "Moore graphs" rather than Moore cages. The term Moore graph was introduced by Hoffman and Singleton [78] in 1960, to honour Edward Moore, who introduced them to the problem of finding graphs that attain the upper bound $M_{\Delta,D}$. At the time Singleton was a PhD student at Princeton University and Hoffman was one of his advisors [104]. Moore and Hoffman came up with a concept to define sets of matrices which are analogous to the tree representation, shown in Figure 3.1, of a graph attaining the bound $M_{\Delta,D}$ [114]. Although, Singleton [114] considered the problem for graphs with even girth and determined the corresponding bound, he only ever used the term Moore graph to refer to graphs with odd girth.

Around the same time and, to the best of my knowledge, independently, the bound for both odd and even girth $M(k,g)$ appeared in the book by Tutte [122]. Since then the bound $M(3,g)$, has been referred to as the "obvious lower bound" [98] and the "very naive bound" by Biggs [27]. Furthermore, since this bound is rarely attained and the number of vertices must be even for cubic graphs Biggs defines the "naive bound" to be $M(3,g)+2$. Biggs [26] also says that a (k,g)-cage with $M(k,g)$ vertices "is said to be a Moore graph if g is odd and a generalized polygon graph if g is even. (The reasons for the apparently bizarre terminology are historical...)".

In his survey on cages Wong [127] states that if $M(k, g) = n_{k,g}$ then "a minimal (k, g)-cage is also called a minimal (k, g)-graph or a Moore graph". Authors of more recent papers [1, 3] have used the term "minimal (k, g)-cage" to mean a (k, g)-cage with order equal to $M(k, g)$. We prefer to use the term Moore cage to indicate a (k, g)-graph of order $M(k, g)$ and hope to avoid any confusion.

Due to the scarcity of Moore cages, research in the area of the degree/girth problem focuses on finding (k, g)-graphs with order smaller than the current best known upper bound and non-existence proofs to increase the lower bound. The term *excess* of G, denoted e, has been introduced to measure how close a (k, g)-graph is to the Moore bound for cages, as follows. Let G be a (k, g)-graph on n vertices and $e = n - M(k, g)$. A summary of known cages and current best known upper bounds on $n(k, g)$ is given in Table 3.1.

k/g	5	6	7	8	9	10	11	12
3	**10**	**14**	**24**	**30**	**58**	**70**	**112**	**126**
4	**19**	**26**	**67**	**80**	275	384		**728**
5	**30**	**42**	152	**170**		1,296	2,688	**2,730**
6	**40**	**62**	294	**312**				**7,812**
7	**50**	**90**		672				32,928
8	80	**114**		**800**				**39,216**
9	96	**146**	1,152	**1,170**			74,752	**74,898**
10	126	**182**		**1,640**				**132,860**
11	156	240		2,618				319,440
12	203	**266**		**2,928**				**354,312**
13	240	336		4,342				738,192
14	288	**366**		**4,760**				**804,468**

Table 3.1: Summary of upper bounds for $n(k, g)$ from [56]. Known (k, g)-cages are shown in bold font.

There have been a number of generalisations of the standard cage question, for example, Harary and Kovács [73] introduced the problem of finding the smallest order of a regular graph with a given girth pair, namely, the odd and even girth of the graph. At a later date, Campbell [37] determined the size of smallest cubic graphs with girth pairs (6,7), (6,9) and (6, 11). More recently, Balbuena, Jiang, Lin, Marcote and Miller [17] established a general lower bound on the order of regular graphs with given girth pair.

The standard cage problem has also been generalised to include non regular graphs. In this case the problem can be stated as finding the minimum order of a graph with degree set $D = \{\delta, \delta + 1, \ldots, \Delta\}$ and girth g. Downs, Gould, Mitchem and Saba [45] determined a lower bound on the order of a graph, in terms of girth, and maximum and minimum degree, as stated in the following theorem and illustrated in Figure 3.3.

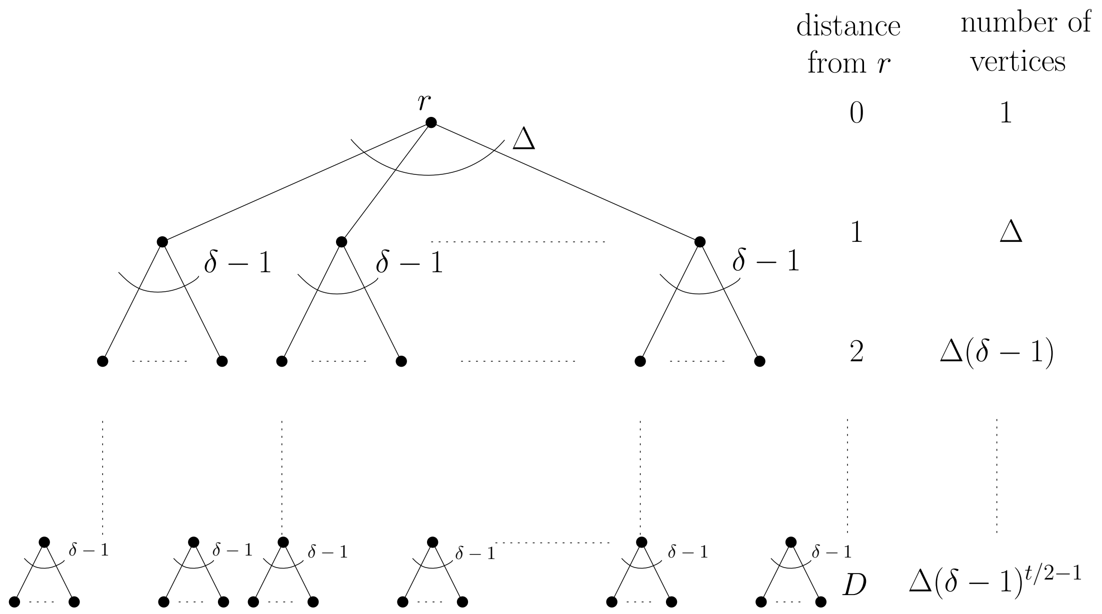


Figure 3.3: Illustration of the lower bound on the order of a graph of even girth $g > 3$.

Theorem 3.1 *[45] Given a graph G, with maximum degree Δ, minimum degree δ and girth $g > 3$,*

$$|V(G)| \geq \begin{cases} 1 + \sum_{i=1}^{k} \Delta(\delta-1)^{i-1} & \text{if } g = 2k+1, \\ 1 + \sum_{i=1}^{(k-1)} \Delta(\delta-1)^{i-1} + (\delta-1)^{(k-1)} & \text{if } g = 2k. \end{cases}$$

Tables of the current best known lower bounds for the degree/girth problem are maintained on the Combinatorics wiki website by Exoo [54].

3.2 Graphs with Maximal Size

In this section, we examine some extremal graph theory problems concerned with determining the maximum possible number of edges or size of a graph of given order and considering some other constraints. Graphs that attain the maximum size under the given constraints are called extremal graphs. We also examine the relationship between the degree/diameter problem, the degree/girth problem and the problem of finding extremal graphs.

In 1941, Turán [46] asked: "How many edges must a graph contain that it should certainly have subgraphs of a prescribed structure?". Alternatively, what is the maximum size of a graph G having order n and the property that if $F \in \mathcal{F}$ then $F \nsubseteq G$. The answer to this question is called the *extremal number*, denoted $ex(n; \mathcal{F})$, and graphs with property $\mathcal{F}$ that have size equal to the extremal number are said to be *extremal graphs*, denoted $EX(n; \mathcal{F})$. Research concerned with this question is referred to as *extremal graph theory.*

A class of problems called *Turán type problems* has evolved around Turán's question. Initial research activity in this area primarily involved finding the maximum size of a graph that does not contain a complete graph as a subgraph. Turán determined the maximum size of a K_{r+1}-free graph and described the construction of such graphs. An n-vertex graph that does not contain any $(r+1)$-vertex clique may be formed by partitioning the set of vertices into r parts of equal or nearly equal sizes, and connecting two vertices by an edge whenever they belong to two different parts thus creating a complete r-partite graph. The resulting graph is known as the Turán graph $T_{n,r}$. Turán's Theorem states that the Turán graph has the largest number of edges among all K_{r+1}-free n-vertex graphs, that is,

$$ex(n; K_{r+1}) = \big(1 - \frac{1}{r}\big)\frac{n^2}{2} \text{ and } EX(n; K_{r+1}) = \{T_{n,r}\}.$$

An earlier result is contained in Mantel's Theorem which states that the maximum size of a triangle-free graph is $\lfloor n^2/4 \rfloor$. This bound is attained by the complete bipartite graphs $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$. Therefore,

$$ex(n; K_3) = \lfloor n^2/4 \rfloor \text{ and } EX(n; K_3) = \{K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}\}.$$

In other words, one must delete nearly half of the edges in K_n to obtain a triangle-free graph. A strengthened form of Mantel's Theorem, due to Bondy [34], states that any graph with at least $n^2/4$ edges must be either the complete bipartite graph $K_{n/2,n/2}$ or it must be pancyclic, that is, the graph contains cycles of all possible lengths up to the number of vertices in the graph.

Turán's Theorem and Mantel's Theorem solve the problem of determining the extremal number $ex(n; K_{r+1})$ and the extremal K_{r+1}-free graphs. Determining the extremal number and enumerating the graphs that are extremal for some other subgraphs $\mathcal{F}$ are still open problems, for example, when $\mathcal{F}$ is the complete bipartite graph $K_{t,u}$, this problem is referred to in the literature as the Zarankiewicz problem and can be stated as follows: let $K_{t,u}$ be the complete bipartite graph on $t+u$ vertices and tu edges, where $t \leq u$. In 1954, Kövári, Sós and Turán [84] determined the lower bound,

$$ex(n, K_{t,u}) \leq c_{t,u} n^{2-\frac{1}{t}}.$$

However, the exact value of $ex(n, K_{t,u})$ is an open problem. Another open problem is finding the extremal number for graphs that do not contain any cycles of a particular length. A general result due to Bollobás [32] states that if $m > 90sn^{1+1/s}$ then the graph has a cycle of length $2s$. Therefore,

$$ex(n; 2s) \leq 90sn^{1+1/s}.$$

In this thesis we focus on another variation of this problem as described in the following section.

3.2.1 Extremal $\{C_3, C_4, \ldots, C_t\}$-free Graphs

In 1975, Erdős [47] posed the problem of finding the maximum size of a graph on n vertices that does not contain three-cycles or four-cycles. We examine a generalised version of this problem, namely, finding the maximum number of edges in a graph of order n that contains no cycle C_k, where $k \leq t$ and $t \geq 3$. We use the notation $ex(n;t) = ex(n;\{C_3, C_4, \ldots, C_t\})$ and the term *extremal number* to indicate this value. Graphs having size equal to the extremal number are called *extremal $\{C_3, C_4, \ldots, C_t\}$-free graphs* or just *extremal graphs* when the context is understood. For given values of n and t the set of extremal graphs is denoted $EX(n;t) = EX(n;\{C_3, C_4, \ldots, C_t\})$.

In contrast to the degree/diameter and degree/girth problems, when considering the problem of determining the extremal number for particular values of n and t, we already know the order of the graph and are concerned with finding the maximum possible size of the graph given some girth constraints. There is however an interesting relationship between the degree/girth problem and the problem of finding extremal graphs. This relationship was expressed by Alon, Hoory and Linial [4] as follows:

> *What is the maximum number of edges in a graph with n vertices and girth g? Put differently, what is the least number of vertices $n = n(d, g)$ in a graph of girth g and an average degree $\geq d$?*

In other words, a graph G that attains the Moore bound for cages $M(k, g)$ is also an extremal $\{C_3, C_4, \ldots, C_{g-1}\}$-free graph $G \in EX(M(k, g), g-1)$ and $ex(M(k, g), g-1) = (M(k, g) \times k)/2$. Furthermore, Alon, Hoory and Linial [4] demonstrated that the Moore bound for cages also holds for graphs that are not degree regular. Alon, Hoory and Linial called this bound the *Moore bound for irregular graphs*, as stated in the following theorem.

Theorem 3.2 *[4]* **(The Moore bound for irregular graphs)** *Any graph with average degree $\overline{d} > 2$ and girth $g \geq 3$ has at least $v(\overline{d}, g)$ vertices, where*

$$v(\overline{d}, g) \geq M(\overline{d}, g) = \begin{cases} 1 + \overline{d} \sum_{i=0}^{\frac{g-3}{2}} (\overline{d} - 1)^i & \text{for odd } g; \\ 2 \sum_{i=0}^{\frac{g-2}{2}} (\overline{d} - 1)^i & \text{for even } g. \end{cases}$$

Information on known cages and what is currently known about the corresponding extremal number is given in Table 3.2. The first column is the order of the (n, k)-cage. The second column contains commonly known names of the corresponding cage. The third column displays the number of cages. The question mark for the (4,7)-cage indicates that it is currently not known if this cage is unique or not. The next three columns indicate the Moore bound for the

(k, g)-cage, if the cage is a Moore cage or not and if the cage is known to be extremal of not. The final column contains the extremal number $ex(n; t)$, where $n = n(k, g)$ and $t = g - 1$. The value of the extremal number, when known, is also given in the final column. When the extremal number is not known then the lower bound on the extremal number due to the corresponding cage is given.

In addition to the (k, g)-cages shown in Table 3.2, all current known cages having girth $g = 6, 8$ or 12, with the exception of the $(7, 6)$-cage, are Moore cages and therefore extremal graphs. The only currently known cages that are known not to be extremal graphs are the (5,5)-cages. All other cages have either been shown to be extremal graphs or provide the current best known lower bound on the corresponding extremal number. Note that the Robertson graph and the McGee graph are unique cages, not Moore cages and yet they are extremal graphs.

$n(k,g)$	Name	#	$M(k,g)$	Moore	Extremal	$ex(n;t)$	
$n(3,5) = 10$	Petersen	1	10	Yes	Yes	$ex(10;4)$	$= 15$
$n(4,5) = 19$	Robertson	1	17	No	Yes	$ex(19;4)$	$= 38$
$n(5,5) = 30$		4	26	No	No	$ex(30;4)$	$= 76$
$n(6,5) = 40$		1	37	No	?	$ex(40;4)$	≥ 120
$n(7,5) = 50$	Hoffman-Singleton	1	50	Yes	Yes	$ex(50;4)$	$= 175$
$n(3,6) = 14$	Heawood	1	14	Yes	Yes	$ex(14;5)$	$= 21$
$n(4,6) = 26$		1	26	Yes	Yes	$ex(26;5)$	$= 52$
$n(5,6) = 42$		1	42	Yes	Yes	$ex(42;5)$	$= 105$
$n(6,6) = 62$		1	62	Yes	Yes	$ex(62;5)$	$= 186$
$n(7,6) = 90$		1	86	No	?	$ex(90;5)$	≥ 315
$n(3,7) = 24$	McGee	1	22	No	Yes	$ex(24;6)$	$= 36$
$n(4,7) = 67$		?	53	No	?	$ex(67;6)$	≥ 134
$n(3,8) = 30$	Tutte-Coxeter	1	30	Yes	Yes	$ex(30;7)$	$= 45$
$n(3,9) = 58$		18	46	No	?	$ex(58;8)$	≥ 87
$n(3,10) = 70$		3	62	No	?	$ex(70;9)$	≥ 105
$n(3,11) = 112$	Balaban	1	94	No	?	$ex(112;10)$	≥ 168
$n(3,12) = 126$	Benson	1	126	Yes	Yes	$ex(126;11)$	$= 189$

Table 3.2: Data used to illustrate the relationship between cages and extremal graphs.

Following the example of Loz, Pérez-Rosés and Pineda-Villavicencio [97] and Exoo [54], we have created a section on the Combinatorics wiki dedicated to the problem of finding extremal $\{C_3, C_4, \ldots, C_t\}$-free graphs [102]. We hope this web site will promote the problem of finding the value of $ex(n; t)$ and the corresponding extremal $\{C_3, C_4, \ldots, C_t\}$-free graphs. We also intend to provide an up-to-date reference for people interested in this problem thus, avoiding any duplication of efforts.

3.3 Connectivity

Menger's Theorem regarding connectivity, appeared in the first book written on graph theory, "Theorie der endlichen und unendlichen Graphen" [83], which was printed in 1936. In order to introduce Menger's theorem we require the following terminology. A $u - v$ *separating set* is a set of vertices $X \subset V(G) - \{u, v\}$ whose removal separates u and v into different components of $V(G) - X$. A collection of $u - v$ paths is said to be *internally disjoint* if they only have the end vertices u and v in common.

Theorem 3.3 (Menger's Theorem 1927) *[105] Let u and v be non-adjacent vertices of a graph G. The minimum number of vertices in a $u - v$ separating set is equal to the maximum number of internally disjoint $u - v$ paths in G.*

Around the same time, Whitney [124] observed the relationship between the vertex connectivity, edge connectivity and minimum degree, namely, $\kappa \leq \lambda \leq \delta$ and developed the following theorem.

Theorem 3.4 (Whitney's Theorem 1932) *[124] A nontrivial graph, G, is k connected for some integer $k \geq 2$ if and only if for each pair u,v of distinct vertices of G, there are at least k internally disjoint $u - v$ paths in G.*

In 1958, Berge [25] asked the question "What is the maximum connectivity of a graph with n vertices and m edges?". This question was answered by Harary [70] in 1962. Harary showed that, for every pair of integers n, m with $2 \leq n - 1 \leq m \leq \binom{n}{2}$, there exists a graph G of order n and size m having vertex connectivity $\kappa = \lfloor 2m/n \rfloor$. Furthermore, it has been shown that a connected graph G having diameter 2, has $\lambda = \delta$ [111].

In addition to the vertex and edge connectivity of a graph, there are a number of other measures of "connectedness" that can be used in the design of fault tolerant networks. One such property is the number of minimum cut sets in the graph. The likelihood of a graph becoming disconnected due to a random failure of components or links increases when the number of minimum cut sets increases. Other such properties consider the number and size of the components in the resulting disconnected graph. In general, it is preferable to have fewer components with all but one component being isolated vertices.

In 1983, in a paper dedicated to Karl Menger, Harary [72] introduced the terms *P-connectivity* and *P-line-connectivity*, denoted respectively by, $\kappa(G{:}P)$ and $\lambda(G{:}P)$ to mean the minimum cardinality of a set of vertices S, respectively, edges F, such that, $G - X$, respectively, $G - F$, is disconnected and every component of $G - X$, respectively, $G - F$, has the property P. The property P may be related to the degree, order, size, diameter or some other property of the resulting components. Harary also used the terms *conditional vertex connectivity* and *conditional vertex connectivity* to mean the same.

The concept of conditional connectivity introduced by Harary considers the properties of the components of $G - X$ or $G - F$. Esfanhanian and Hakimi [52] introduced the notion of *restricted connectivity*, where the restrictions are on the cut set rather than on the resulting connected components. The motivation for this concept was to be able to consider heterogeneous networks in which the likelihood of failure is vastly different depending upon the nodes or edges being considered, for example, when considering the edges or links in a computer network, it is less likely that a fiberoptic cable will fail compared to a satellite link. Analogously, when considering the nodes of the network to be computers and web servers, large corporations will have back-up web servers for redundancy in case a server may fail, so from the network point of view the node is not disconnected. Formally, the *R-edge connectivity* of a graph G, denoted $\lambda(G|F{:}R)$ is the minimum cardinality of a set of edges F, such that $G - F$ is disconnected and F is restricted to a given subset of edges R, that is, $F \subseteq R \subseteq E(G)$. The *R-vertex connectivity* of a graph G, denoted $\kappa(G|X{:}R)$, is defined similarly with $X \subseteq R \subseteq V(G)$. Esfanhanian and Hakimi [52] describe an algorithm for computing the P-edge-connectivity of a graph but conjecture that computing the restricted connectivity of a graph, with the restriction being that the cut set is nontrivial, is NP-hard. Furthermore, $\kappa(G|X{:}R)$, when the restriction is that X does not contain all the vertices that are adjacent to one vertex, does not exist for some graphs, for example, see Figure 3.4. For these reasons, it is desirable to be able to determine the connectivity of a graph using our knowledge of other parameters that are more easily calculated, for example diameter, girth and maximum and minimum degree.

Hellwig and Volkmann [76] use the notation $\lambda'(G) = \lambda'$ to mean the minimum cardinality of an edge cut set over all edge cuts F, such that $G - F$ does not contain any isolated vertices. They call a graph λ'-optimal, if $\lambda'(G) = \xi(G)$, where $\xi(G)$ is the minimum edge degree in G.

Hellwig, Rautenbach and Volkmann [75] introduced $\lambda_p(G)$ defined as the minimum cardinality of an edge cut set over all edge cut sets F, such that each component of $G-F$ contains at least p vertices. In the same paper, a more general parameter $\lambda_{pq}(G)$, is defined as the minimum cardinality of an edge cut set over all edge cut sets F, such that one component of $G - F$ contains at least p vertices and another component of $G-F$ contains at least q vertices, where p and q are positive integers.

The notion of superconnectedness was proposed in [23, 30, 31]. A graph is *superconnected*, for short *super-κ*, if all minimum vertex cut sets are trivial, see Boesch [30], Boesch and Tindell [31] and Fiol, Fàbrega and Escudero [62]. Observe that a superconnected graph is necessarily maximally connected, $\kappa = \delta$, but the converse is not true, for example, a cycle C_n, where $n \geq 6$, is maximally connected but not superconnected. A cut set X of G is called a *nontrivial cut set* if X does not contain the neighbourhood $N(u)$ of any vertex $u \notin X$. Provided that some nontrivial cut set exists, the *superconnectivity* of G denoted by κ_1 was defined in [10, 62] as follows,

$$\kappa_1 = \kappa_1(G) = \min\{|X| : X \text{ is a nontrivial cut set}\}.$$

A nontrivial cut set X is called a *κ_1-cut* if $|X| = \kappa_1$. Notice that if $\kappa_1 \leq \delta$, then $\kappa_1 = \kappa$ and $\kappa_1 > \delta$ is a sufficient and necessary condition for G to be super-κ, since all the minimum disconnecting sets with cardinality equal to δ must be trivial. A *nontrivial edge cut*, the *edge-superconnectivity* $\lambda_1 = \lambda_1(G)$ and a *λ_1-cut* are defined analogously. The index $\lambda_1(G)$ is also known under the name *restricted edge connectivity* which was introduced by Esfahanian and

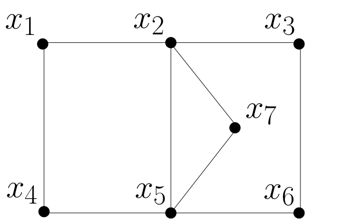


Figure 3.4: A superconnected graph.

Hakimi [52], who denoted it by $\lambda'(G)$. The existence of restricted edge cuts has been studied in [36, 110, 128, 130].

Sufficient conditions for maximally connected dense graphs in terms of the diameter and girth of a graph were given by Soneoka, Nakada, Imase and Peyrat [117]. These conditions are summarised in the following theorem.

Theorem 3.5 *[117] Let δ, κ, λ, D, g be respectively the minimum degree, vertex-connectivity, edge-connectivity, diameter and girth of a graph G.*

$$\lambda = \delta \text{ if } \begin{cases} D \leq g-1, & g \text{ odd,} \\ D \leq g-2, & g \text{ even.} \end{cases}$$

$$\kappa = \delta \text{ if } \begin{cases} D \leq g-2, & g \text{ odd,} \\ D \leq g-3, & g \text{ even.} \end{cases}$$

Balbuena, Carmona, Fàbrega and Fiol improved upon Theorem 3.5 as shown in Theorem 3.6.

Theorem 3.6 *[11] Let G be a graph with minimum degree $\delta \geq 2$, girth g, edge minimum degree ξ, connectivities λ and κ. Then*

$$\lambda \geq \min\{\delta, 4\} \text{ if } D \leq g-1, g \text{ even}$$

$$\kappa \geq \min\{\delta, 4\} \text{ if } D \leq g-2, g \text{ odd}$$

The sufficient conditions given in Theorem 3.6 have been improved by Balbuena and Marcote [18] for regular graphs as shown in the following theorem.

Theorem 3.7 *[18] Let G be an r-regular graph with $r \geq 2$ girth g and connectivities λ and κ. Then*

(i) $\lambda \geq 2$ *if* $D \leq 2\lfloor (g-1)/2 \rfloor + 2$.
(ii) $\kappa \geq 2$ *if any of the following statements hold:*

$D \leq 2\lfloor (g-1)/2 \rfloor + 2$ *when* $r \leq 3$.
$D \leq 2\lfloor (g-1)/2 \rfloor + 1$.

(iii) $\kappa \geq \min\{r, 3\}$ *if* $D \leq g-1$.

(iv) $\kappa \geq \min\{r, 6\}$ *if* $D \leq g-2$ *for even* g.

Analogously, Fàbrega and Fiol [58] determined sufficient conditions, in terms of girth and diameter, for a graph to be super-κ or super-λ connected. These conditions are presented in the following theorem.

Theorem 3.8 *[58] Let D and g be respectively the diameter and girth of a graph G.*

$$G \text{ is super-}\lambda \text{ if} \begin{cases} D \leq g-2, & g \text{ odd}, \\ D \leq g-3, & g \text{ even}. \end{cases}$$

$$G \text{ is super-}\kappa \text{ if} \begin{cases} D \leq g-3, & g \text{ odd}, \\ D \leq g-4, & g \text{ even}. \end{cases}$$

Subsequently, Balbuena, García-Vázquez and Marcote [16] and Balbuena, Cera, Diánez, García-Vázquez, and Marcote [13] refined these results as demonstrated by the following theorems.

Theorem 3.9 *[16] Let G be a graph with minimum degree $\delta \geq 2$, diameter D, girth g, edge minimum degree ξ, and edge superconnectivity λ_1. Then,*

$$\lambda_1 = \xi \text{ if } D \leq g-2.$$

Theorem 3.10 *[13] Let G be a graph with minimum degree $\delta \geq 2$, diameter D, girth g, edge minimum degree ξ, and superconnectivity κ_1. Then,*

$$\kappa_1 \geq \xi \text{ if } D \leq g-3.$$

In [77] Hellwig and Volkmann provided a comprehensive survey of sufficient conditions for a graph to achieve lower bounds on κ, λ, κ_1 and λ_1.

3.3.1 Connectivity of Moore Graphs

The complete graphs, $K_{\Delta+1}$, are, by definition, maximally connected and superconnected. The cycles C_{2D+1}, are maximally connected but not super-λ or super-κ connected. The Petersen graph, the Hoffman-Singleton graph and the Moore graph with $\Delta = 57$ and diameter 2, if it exists, all have diameter 2 and odd girth 5. Application of Theorem 3.5 determines that these graphs are maximally connected. Similarly, applying Theorem 3.8 asserts that the graphs are super-κ.

In summary, all Moore graphs are maximally connected. Furthermore, all Moore graphs, with the exception of the cycles C_{2D+1}, are superconnected.

3.3.2 Connectivity of Cages

In 1997, Fu, Huang and Rodger [64] proved that all (k, g)-cages are 2-connected and that all $(3, g)$-cages are 3-connected. They further conjectured that every (k, g)-cage is k-connected. In support of this conjecture, Jiang and Mubayi [81] and, independently, Daven and Rodger [43], proved that every (k, g)-cage with $k \geq 3$ is 3-connected. Xu, Wang and Wang [129] showed that all $(4, g)$-cages are 4-connected. The fact that every (k, g)-cage with $k \geq 4$ and $g \geq 10$ is 4-connected was proven by Marcote, Balbuena, Pelayo and Fàbrega [100]. It has also been shown by Marcote, Balbuena and Pelayo [101] that every (k, g)-cage is maximally connected for girth $g = 5, 6$ and 8. Publications by Lin, Miller and Balbuena [94] and Lin, Balbuena, Marcote and Miller [91] contain lower bounds on the vertex connectivity that support the conjecture that every (k, g)-cage is k-connected, however, the conjecture remains open. More recently, Lin, Lu, Wu and Yu [92] showed that $(4, g)$-cages with even girth, $g \geq 12$, are superconnected.

On the other hand, the problem of determining the edge connectivity of cages has been settled. All (k, g)-cages have been proven to be k-edge connected if g is odd, by Wang, Xu and Wang [126] and for even girth by Lin, Miller and Rodger [93]. Moreover, the fact that all (k, g)-cages are edge superconnected was proven by Marcote and Balbuena [99], for odd girth and Lin, Miller, Balbuena and Marcote [95] for even girth.

3.3.3 Connectivity of Extremal Graphs

It is well known that for $n \leq t$ the extremal graphs are the trees on n vertices. Knowledge of the structure of the extremal graphs for $n \leq t$ allows us to determine that these graphs are maximally connected with connectivity $\kappa = 1$. Furthermore, since $D \leq t - 1 \leq g - 2$, for all extremal graphs, applying Theorem 3.8 tells us that the extremal graphs are maximally connected, that is, $\kappa = \lambda = \delta$ for even t.

Tang, Lin, Balbuena and Miller [118] further investigated the connectivity of extremal graphs and assert that, for every extremal graph G, the restricted edge connectivity is equal to the minimum edge degree, that is, $\lambda'(G) = \xi(G)$ as stated in the following theorem.

Theorem 3.11 *[118] Let G be an extremal graph $G \in EX(n, t)$, for $n \geq 6$, minimum degree $\delta \geq 2$ and minimum edge degree $\xi = \xi(G) = min\{deg(u) + deg(v) - 2 : uv \in E(G)\}$. Then G has a restricted edge connectivity $\lambda' = \xi$.*

In summary, all extremal graphs having minimum degree $\delta \geq 2$ and odd girth are super edge connected and maximally connected when t is even. Tang, Lin, Balbuena and Miller [118] conjectured that extremal graphs are also maximally connected for odd t. In order to determine if extremal graphs are maximally connected for $n > t$ we require further knowledge of their structural properties. In particular, knowledge of the girth, diameter, and maximum and minimum degree of a graph is helpful in determining the connectivity. In Section 4.2, we summarise what is currently known about the structure of extremal graphs.

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

Paul Richard Halmos (1916-2006) I Want to Be a Mathematician

4

Extremal Graphs

In this chapter, we give a brief history of research in the area of extremal $\{C_3, C_4, \ldots, C_t\}$-free graphs. Furthermore, we summarise what is currently known about extremal values of $ex(n;t)$ including our new results which are indicated by the symbol $\diamondsuit$. In Section 4.1, we make some general observations and introduce the notation that we use throughout this chapter. In Section 4.2, we list some known structural properties of extremal graphs. Using the knowledge of the observations given in Section 4.1 and the structural properties described in Section 4.2 we have developed an algorithm that we use to generate a number of graphs with size greater then the current best known lower bounds on $ex(n;t)$, for $n \leq 200$ and $4 \leq t \leq 11$. We call this algorithm the Grow and Prune or GAP algorithm. In Section 4.3, we describe our GAP algorithm. The new lower bounds on $ex(n;t)$ due to graphs produced by our algorithm are given in subsequent sections.

In Section 4.4, we give a summary of the current best known lower bounds and exact values of $ex(n;t)$, for $n \leq 200$ and $t = 3$ and 4, drawing from the pioneering papers by Garnick and Neuwejaar [68] and Garnick, Kwong and Lazebnik [66,67] and recent work by Abajo, Balbuena and Diánez [1]. Furthermore, we use our GAP algorithm to improve the lower bounds on $ex(n;4)$ given in [1,66,68], for 120 different values of $n \leq 200$.

In Section 4.5, we give a number of constructions that produce new infinite families of graphs which we prove to be extremal. In Section 4.6, we give a summary of the current best known lower bounds and exact values of $ex(n;t)$, for $t = 5, 6$ and 7, including our new results. Moreover, we provide full proofs, for $ex(30;6) = 47$, $ex(31;6) = 49$ and $ex(32;6) = 51$. In Section 4.7, we summarise what is known about $ex(n;t)$, for $t = 8, 9, 10$ and 11, and use our GAP algorithm to generate new lower bounds on $ex(n;t)$, for $n \leq 200$ and $t = 8, 9, 10$ and 11. Furthermore, we establish the exact values of the extremal numbers: $ex(n;8)$, for $n = 23, 24, 25, 26$; $ex(n;9)$, for $n = 26, 27, 28, 29$; and $ex(127;11)$.

4.1 Observations and Notation

In 1975, Erdős [47] posed the problem of finding the maximum size of graphs that do not contain three-cycles or four-cycles. In this chapter, we examine a generalised version of this problem, namely, finding the maximum number of edges in a graph of order n that contains no cycle C_k, where $k \leq t$ and $t \geq 3$. We use the notation $ex(n;t) = ex(n;\{C_3, C_4, \ldots, C_t\})$ and the term *extremal number* to indicate this value. Graphs having size equal to the extremal number are called *extremal* $\{C_3, C_4, \ldots, C_t\}$*-free graphs* or just *extremal graphs* when the context is understood. Given n and t, the set of extremal graphs is denoted $EX(n;t) = EX(n;\{C_3, C_4, \ldots, C_t\})$. Furthermore, we use the notation $ex_l(n;t)$ and $ex_u(n;t)$ to denote current best known lower and upper bounds on $ex(n;t)$ when the extremal number is not yet known. The sets of graphs that demonstrate the bounds $ex_l(n;t)$ and $ex_u(n;t)$ are denoted by $EX_l(n;t)$ and $EX_u(n;t)$, respectively.

Since, by definition, extremal graphs do not contain any cycles of length less than or equal to t, the extremal graphs, for $n \leq t$, must be acyclic. Therefore, for $n \leq t$, the extremal graphs are the trees on n vertices, T_n, including the stars $K_{1,n-1}$ and paths P_n, and $ex(n;t) = n-1$. Since $EX(n;t) = \{T_n\}$, for $n \leq t$, the problem of enumerating the extremal graphs in the set $EX(n;t)$ is equivalent to enumerating the trees T_n (see [71] pages 178-192). For $n = t+1$, the cycles on n vertices, C_n, are the unique extremal graphs, that is, $EX(t+1;t) = \{C_{t+1}\}$, and $ex(t+1;t) = t+1$. For $t+1 < n \leq 3t/2$, the extremal number is $ex(n;t) = n$ and the set of extremal graphs $EX(n;t)$ consists of the unicyclic graphs on n vertices.

Extremal graphs that do not contain cycles of length less than or equal to $t+1$, by definition, are free of cycles of length t or less, therefore, the function $ex(n;t)$ is non-increasing in t for fixed n. More formally,

$$ex(n;t+1) \leq ex(n;t). \tag{4.1}$$

Given $G \in EX(n;t)$ and a vertex $x \notin V(G)$, a new graph G' with vertex set $V(G') = V(G) \cup \{x\}$ and edge set $E(G') = E(G) \cup \{xy\}$, where $y \in V(G)$, can be obtained. Therefore,

$$ex(n+1;t) \geq ex(n;t) + 1. \tag{4.2}$$

Let the minimum and maximum degrees be denoted by δ and Δ respectively. If $\overline{d}$ denotes the average degree, then

$$\delta \leq \lfloor \overline{d} \rfloor \leq \overline{d} \leq \lceil \overline{d} \rceil \leq \Delta. \tag{4.3}$$

Given a graph $G \in EX(n;t)$, removing a vertex v of degree $deg(v) = \delta$ results in a graph of order $n-1$, girth $g \geq t$ and size $|E(G-v)| \leq ex(n-1;t)$. Therefore,

$$ex(n;t) - \delta \leq ex(n-1;t). \tag{4.4}$$

Combining the Inequalities 4.4 and 4.3 gives the following inequality.

$$ex(n;t) - ex(n-1;t) \leq \delta \leq \lfloor 2ex(n;t)/n \rfloor \leq \lceil 2ex(n;t)/n \rceil \leq \Delta \tag{4.5}$$

Inequality 4.4 is the result obtained by the removal of a vertex of minimum degree from an extremal graph $G \in EX(n;t)$. As a natural extension, consider the removal of any vertex or set of vertices from $ex(n;t)$. Given a graph $G \in EX(n;t)$, removing a vertex v results in a graph $G - v$ of order $n-1$, girth $g \geq t+1$ and size $|E(G)| - deg(v)$. Consequently, $ex(n;t) - deg(v) \leq ex(n-1;t)$. Furthermore, deleting a path $v_1, v_2, \ldots, v_k$, where $k \leq t$, from G will delete k vertices and $deg(v_1) + deg(v_2) + \ldots + deg(v_k) - k + 1$ edges. Therefore,

Observation 4.1 *For $G \in EX(n;t)$ and $P_k \subset G$*

$$|E(G - P_k)| = ex(n;t) - deg(x_1) - deg(x_2) - \ldots - deg(x_k) + k - 1 \leq ex(n-k;t)$$

We adopt the notation $T_{\Delta,\delta,t}$ to denote the tree having height $\lfloor (t+1)/2 \rfloor$ and root r such that $deg(r) = \Delta$, the leaves have degree 1, and every other vertex $v \neq r$ has $deg(v) = \delta$. Since $g \geq t+1$, every $G \in EX(n;t)$ must contain $T_{\Delta,\delta,t}$ as a subgraph. Furthermore, the order of $T_{\Delta,\delta,t}$ is equal to the lower bound on the order of a graph, in terms of girth, maximum and minimum degree, as stated in Theorem 3.1. In the following theorem we restate Theorem 3.1 in terms of t.

Theorem 4.1 *[45] Let G be a graph with maximum degree Δ, minimum degree δ and girth $g > t > 3$. Then G contains the tree $T_{\Delta,\delta,t}$ as a subgraph. Thus,*

$$|V(G)| \geq |V(T_{\Delta,\delta,t})| = \begin{cases} 1 + \sum_{i=1}^{t/2} \Delta(\delta-1)^{i-1} & \text{for even } t, \\ 1 + \sum_{i=1}^{(t-1)/2} \Delta(\delta-1)^{i-1} + (\delta-1)^{(t-1)/2} & \text{for odd } t. \end{cases}$$

Moreover, $|E(T_{\Delta,\delta,t})| = |V(T_{\Delta,\delta,t})| - 1$. We use X to denote the set of vertices $X = V(G - T_{\Delta,\delta,t})$. We use the notation $T_{\Delta,\delta,t}(\mathcal{F})$ to represent the tree $T_{\Delta,\delta,t}$ with additional forbidden subgraph constraints, that is, if $F \subset \mathcal{F}$ then $F \not\subseteq T_{\Delta,\delta,t}$. We use the terms *short cycle* and *forbidden cycle* to refer to a cycle C_k, where $k \leq t$, these cycles are, by definition, forbidden subgraphs and will not be explicitly listed in the set $\mathcal{F}$.

Abajo and Diánez [3] observed that in order to prove $ex(n;t) \leq m$ it is sufficient to prove the non-existence of a graph of order $m+1$ and girth $g > t$. In order to prove the non-existence of a graph $G \in EX(n;t)$ of order $m+1$ we determine a number of necessary structural properties of G. In particular, we use Observation 4.1 to determine that particular paths are forbidden in G. We use the notation P_k^j to mean the path on k vertices such that all k vertices in the path have degree j in G.

4.2 Structural Properties of Extremal Graphs

Knowledge of the structural properties of extremal graphs can be utilised to determine the values of currently unknown extremal numbers and the existence of corresponding extremal graphs. In particular, we exploit knowledge of the diameter and girth of extremal graphs in our GAP, in order to discover new and improved lower bounds on $ex(n;t)$, for $n \leq 200$ and $3 < t < 11$. Furthermore, increased understanding of the structure of extremal graphs may provide some insight to Tang, Lin, Balbuena and Miller's [118] conjecture that all extremal graphs are maximally connected. In Sections 4.2.1 and 4.2.2, we provide a summary of what is currently known about the diameter and girth of extremal graphs.

4.2.1 The Diameter of Extremal Graphs

Garnick, Kwong and Lazebnik [66] investigated the diameter of the extremal graphs $G \in EX(n;4)$ and determined that the diameter is at most 3. The same authors observed that if $d(x) = \delta(G) = 1$ then the graph $G - \{x\}$ has diameter 2. Recent work by Balbuena, Cera, Diánez and García-Vázquez [14] generalised these results for other values of t by proving that for $G \in EX(n;t)$ the diameter $D \leq t-1$ and if $d(x) = \delta(G) = 1$ then the graph $G - \{x\}$ has diameter $D \leq t-2$. Since $t < g$, any extremal graph $G \in EX(n;t)$ has $D < t < g$ therefore $D \leq g-2$. Analogously, results for cages were determined by Erdős and Sachs [50], who proved that if D is the diameter of a (k,g)-cage, then $D \leq g$. All current known cages and graphs that demonstrate the current best known upper bound on the order of a (k,g)-cage have $D < g$. Furthermore, if a (k,g)-graph is a Moore cage then $D = \lfloor (g-1)/2 \rfloor$ [26].

4.2.2 The Girth of Extremal Graphs

Garnick and Nieuwejaar [68] asked the question "Is there a constant c, such that, for all $n \geq ck$, the girth of an extremal graph $G \in EX(n;t)$ is $g = t+1$?". This question has been the subject of much of the research in the area of extremal graphs. Lazebnik and Wang [90] determined a lower bound on n in terms of t that ensures that the girth of a graph $G \in EX(n, t \geq 12)$ is $g = t+1$, namely, $n \geq 2^{a^2+a+1}t^a$, where $a = t - 3 - \lfloor (t-2)/4 \rfloor$. Balbuena and García-Vázquez [15] recently improved this lower bound to $n > (2(t-2)^{t-2} + t - 5)/(t-3)$, for $G \in EX(n, t \geq 6)$.

Lazebnik and Wang [90] showed that $ex(2t+2;t) = 2t+4$, for $t \geq 12$, and there exists $G \in EX(2t+2;t)$ with $g = t+2$. Balbuena, Cera, Diánez and García-Vázquez [14] showed that, for $n \leq t + 1 + \lfloor (t-2)/2 \rfloor$, there are extremal graphs with $g = t+2$. We have found there are no extremal graphs $G \in EX(n;t)$ with girth $g = t+1$ and minimum degree $\delta \geq 2$, for $t+2 \leq n \leq t+1+\lfloor (t-2)/2 \rfloor$, contradicting item (*iii*) of the following theorem.

Theorem 4.2 *[90] Let $G \in EX(n;t)$ then, for $n \geq t+1$ and $t \geq 3$*

(i) there exists an extremal graph with $\delta \geq 2$;
(ii) there exists an extremal graph of girth $t+1$;
(iii) if $n \neq t+2$ there exists an extremal graph with minimum degree $\delta \geq 2$ and girth $g = t+1$.

The extremal graphs $G \in EX(n;t)$, for $t+2 \leq n \leq t+1+\lfloor (t-2)/2 \rfloor$, are unicyclic, for example, $G \in EX(11;8)$ has $n = 11 \geq t+1 = 9$, $t = 8 \geq 3$ and $n \neq t+2$. The graphs in the set $EX(11;8)$ include: the cycle C_{11}; the cycle C_{10} with a pendant vertex; and the cycle C_9 with either two pendant vertices or a path of length 2 attached (see Figure 4.1). There is no graph $G \in EX(11;8)$ with girth $g = 9$ and minimum degree $\delta \geq 2$. However, if we change the assumption to $n > t+1+\lfloor (t-2)/2 \rfloor$ then the proof of Lazebnik and Wang [90] holds and the corrected version of the Theorem 4.2 (*iii*) can be stated as follows.

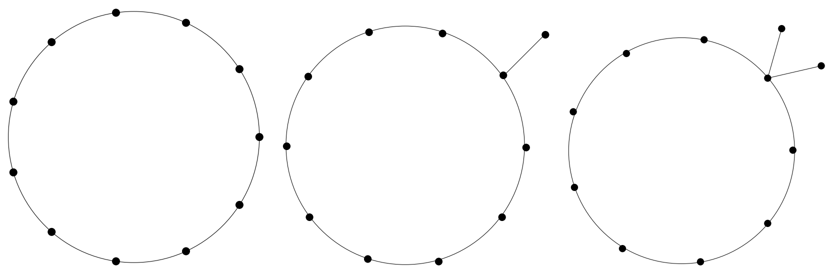


Figure 4.1: Three different graphs in $EX(11;8)$.

$\diamondsuit$ **Theorem 4.3** *Let $G \in EX(n;t)$, for $t \geq 3$. Then there exists an extremal graph with minimum degree $\delta \geq 2$ and girth $g = t+1$, for $n > t+1+\lfloor (t-2)/2 \rfloor$.*

A summary of known conditions under which the girth of any extremal graph $G \in EX(n;t)$ is known to be $t+1$ and upper bounds on the girth are given in the following two lists.

Conditions under which the girth of any extremal graph $G \in EX(n;t)$ is known to be $t+1$.

- $G \in EX(n \geq 7;4)$ [68]
- $G \in EX(n \geq 8;5)$ [90]
- $G \in EX(n \geq 12;6)$, for $n \notin \{15, 80, 170\}$ [14]
- $G \in EX(n \geq 16;6)$ [1]
- $G \in EX(n \geq 783;7)$ [15]
- $G \in EX(n \geq t+5;t)$, when $D = t-1$ and $\delta \geq 3$ [14]
- $G \in EX(n;t)$, when $\Delta = t+1$ [90]

For upper bounds on the girth of an extremal graph, we have

- $g \leq t+2$, for $n > (2(a-2)^{t-2} + a - 5)/(a-3)$, where $a = \lceil (t+1)/2 \rceil \geq 4$ [15]
- $g \leq 2t-4$, for $n \geq 2t-2$ and $t \geq 5$ [15]
- $g \leq 2t-5$, for $n \geq 2t-3$ and $t \geq 7$ [14]

4.3 Grow and Prune Algorithm

In this section we describe the motivation and philosophy behind the development of our GAP algorithm. Furthermore, we provide a description of how the algorithm is used to "Grow" and "Prune" a graph. Pseudocode for our GAP algorithm is contained in Appendix A.

Motivated by the fact that a number of cages are extremal graphs (see Table 3.2) we decided to examine the (k, g)-graphs that give the current best known upper bound on the order of a (k, g)-cage for the degree/girth problem (see Table 3.1). The graphs that attain these bounds and descriptions of the construction techniques used to create these graphs are given in the dynamic cage survey by Exoo and Jajcay [56]. We found that the (k, g)-graphs that give the best known upper bounds on the order $n(k, g)$ for the degree/girth problem often give the current best known lower bound on $ex(n(k, g), g-1)$. One notable exception being the $(5, 5)$-cages which have order 30 and one edge less than the corresponding extremal number $ex(30; 4) = 76$.

In order to take advantage of these graphs we developed our GAP algorithm which is essentially a greedy algorithm. The algorithm takes as input: a graph which is extremal, or provides a good lower bound on the extremal number; an integer t; and an array of integers that are the current best known lower bounds on $ex(n; t)$, for $n \leq 200$. The input or *seed* graph is "Pruned" by deleting the vertex with lowest degree and "Grown" by grafting edges and paths onto the original graph in such a manner that the girth is maintained. Whenever the current best known lower bound is improved the array is updated.

All current known cages and graphs that demonstrate the current best known upper bound on the order of a (k, g)-cage have $D < g$. For this reason the GAP algorithm was designed with the assumption that the input graph is degree regular, with girth $g = t+1$ and diameter $D < t+1 \leq g$. We have, however, run the GAP algorithm using non regular graphs as input with pleasing results, for example, when using the subdivided Petersen graph as a seed graph, the GAP algorithm produced two new extremal graphs (see Theorems 4.23 and 4.24).

4.3.1 Grow

Given a graph $G_l \in EX_l(n; t)$ we can obtain a lower bound on $ex(n+1; t)$ by application of Observation 4.2, that is, $ex_l(n+1; t) \geq ex_l(n; t) + 1$. Therefore, we can always "Grow" a graph G_l by adding a pendant vertex thus increasing the order and size by one while maintaining the girth. In some cases we can do better than this, for example, if the diameter of G_l is equal to t then we can add an edge between two vertices in G_l that are at distance t apart without increasing the number of vertices in the graph or violating girth constraints. In order to take advantage of this we add pendant vertices in such a fashion that the diameter of the resulting graph is maximised.

More formally, the GAP algorithm "Grows" a graph G by selecting two arbitrary vertices u and v whose distance from each other is equal to the diameter of G, that is, $d(u,v) = D$. If $D = t$, then we add the edge $\{u, v\}$, otherwise we add the path $\{\{u, n{+}1\}, \{n{+}1, n{+}2\}, \ldots, \{n{+}k, v\}\}$, where $n+1, n+2, \ldots, n+k$ are all new vertices in G. At the end of each iteration of the Grow algorithm we have either a graph G' with $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)| + 1$ or a graph G'' with $|V(G'')| = |V(G)| + k$ and $|E(G')| = |E(G)| + k + 1$.

An alternative implementation of the "Grow" algorithm is to subdivide an edge of G_l. We experimented with the subdivision method but the results that we obtained were either equivalent or inferior to those obtained by our GAP algorithm.

4.3.2 Prune

Our GAP algorithm "Prunes" a graph G in two steps. The first step consists of finding g vertices that lie on a girth cycle of G and deleting them one by one. The second step consists of finding a vertex $v \in G$, such that, $deg(v) = \delta$ and deleting it. This step is then iterated until $|V(G)| = t + 1$. In some cases we ran the second step of the algorithm manually and improved the results by carefully selecting the next vertices to be deleted, for example, after deleting all of the vertices that were on a girth cycle $C_g \subset G$ we would then delete all vertices in another cycle $C_k \in G$ such that $|V(C_k) \cap V(C_g)|$ is maximal.

The GAP algorithm is very simple and fast in comparison to hill-climbing and back-tracking techniques used by Garnick, Kwong and Lazebnik [66] and the hybrid simulated annealing and genetic algorithm used by Tang, Lin, Balbuena and Miller [118]. The greedy algorithm developed by Abajo and Diánez [3] is similar to our algorithm except that they use a combination of subdividing arbitrary edges and adding paths.

Considering the simplicity of our algorithm we were pleased that the lower bounds produced by the GAP algorithm, in most cases, improved upon or, at least, matched those given in [3,66,118]. Admittedly the performance of our GAP algorithm is heavily dependent upon good seed graphs.

4.3.3 Seed Graphs

The seed graphs that we used for the GAP algorithm include: the cycles C_n; the (k, g)-cages; the (k, g)-graphs that give the current best known upper bound on $n(k, g)$; and the graphs presented in Section 4.5.

With the exception of the (7,6)-cage and the cages that are point-line incidence graphs of generalised polygons, the adjacency lists for the known cages were available to us in the Maple software package. We acquired the adjacency list for the (7,6)-cage from Exoo [53]. We constructed: the (k,6)-cages, for $k = 5, 6, 8, 9, 10$ using the projective planes, for order $k-1$; the (k,8)-cages, for $k = 3, 4, 5$ using the generalised quadrangles of order (2,2), (3,3) and (4,4), respectively, and; the Benson graph or (3,12)-cage which is the incidence graph of the generalised hexagon of order (2,2). We used the list of projective planes and generalised polygons on the website maintained by Moorhouse [107] to do this.

We acquired adjacency lists for graphs that give the current best known upper bounds for the order of (k, g)-cages from Exoo [53, 55].

In some cases, we found graphs that provide the current best known lower bounds, but we did not acquire the adjacency list or construct the graphs, for example, we considered all the constructions listed in dynamic cage survey by Exoo and Jajcay [56] and the trivalent symmetric graphs listed by Condor and Dobcsányi [40]. In such cases we took advantage of knowledge of the structure of these graphs, for example, degree regularity and diameter to calculate new lower bounds manually.

4.4 $ex(n; t)$, for $t \in \{3, 4\}$

The problem of finding $ex(n; 3)$ and corresponding extremal graphs was solved, in 1907, by Mantel's Theorem which states that the maximum size of a triangle-free graph is $\lfloor n^2/4 \rfloor$. This bound is attained by the complete bipartite graphs $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$. Therefore,

$$ex(n; K_3) = \lfloor n^2/4 \rfloor \text{ and } EX(n; K_3) = \{K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}\}.$$

In 1975, Erdős [47] posed the problem of finding the maximum size of graphs that do not contain three-cycles or four-cycles and conjectured that $ex(n; 4) = (1/2 + O(1))^{3/2} n^{3/2}$. The current best known lower bound on $ex(n; 4)$ to date is due to Garnick and Neuwejaar [68], namely,

$$\frac{1}{2\sqrt{2}} n^{3/2} \leq ex(n; 4)$$

The best known upper bound to date is due to Garnick, Kwong and Lazebnik [66], namely,

$$ex(n; 4) \leq \frac{1}{2} n \sqrt{n-1}$$

This inequality is an equality if and only if G is a Moore graph of diameter 2 or an isolated vertex. We use this result to generate the upper bounds on $ex(n;4)$ that are listed in Table B.1 in Appendix B.

Garnick, Kwong and Lazebnik [66] showed that the diameter of $G \in EX(n;4)$ is at most 3 and if $d(x) = \delta(G) = 1$ then the graph $G - \{x\}$ has diameter at most 2. Using this result they were able to draw from the work of Bondy, Erdős and Fajtlowicz [35] to show that the extremal graphs $G \in EX(n;4)$ include:

- The stars $K_{1,n-1}$, for $n \leq 4$
- The Moore graphs
  - C_5
  - The Petersen graph
  - The Hoffman-Singleton graph
  - The Moore graph with girth 5 and degree 57 if it exists
- The polarity graphs (see Section 2.2)

Proposition 4.1 *[66] For all $G \in EX(n;4)$ of order $n \geq 1$,*

(i) $n \geq 1 + \Delta\delta \geq 1 + \delta^2$.
(ii) $\delta \geq ex(n;t) - ex(n-1;t)$ *and* $\Delta \geq \lceil 2ex(n;4)/n \rceil$.
(iii) $n \geq 1 + \lceil 2ex(n;4)/n \rceil (ex(n;4) - ex(n-1;4))$.

The maximum size of graphs that do not contain three-cycles or four-cycles, $ex(n;4)$, for $n \leq 24$, and constructive lower bounds, for $n \leq 200$, are given by Garnick, Kwong and Lazebnik in [66]. The constructive lower bounds were produced by algorithms that use hill-climbing and back-tracking techniques. Additional values of $ex(n;4)$, for $25 \leq n \leq 30$ were determined by Garnick and Nieuwejaar in [68]. These results are summarised in Table 4.1. The values that are displayed in bold font are exact values of the extremal number, all other values are lower bounds on $ex(n;4)$.

The results in Table 4.1 were not improved upon for over ten years. In 2010, Abajo, Balbuena and Diánez [1] constructed some infinite families of graphs that improve the lower bounds on $ex(n;4)$. The new lower bounds produced by these constructions are given in Theorem 4.4, where these constructions are improvements on the previously known lower bounds on $ex(n;4)$ they are displayed in Table 4.2.

Theorem 4.4 *[1] Let $f_4(n) = ex(n;4)$ and let $q \geq 3$ be a prime power. Then*

(i) $f_4(2q^2 + q) \geq q^2(q+1) + (q+1)f_4(q)$.
(ii) $f_4(2q^2 + q - h) \geq q^2(q+1) + (q+1)f_4(q) - hq + \epsilon$, *for* $q = 8, 9, 11$, *where* $\epsilon = -h$, *for* $h = 1, 2$, *and* $\epsilon = -2$, *for* $h = 3, ..., q+1$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**5**	**6**	**8**	**10**	**12**
10	**15**	**16**	**18**	**21**	**23**	**26**	**28**	**31**	**34**	**38**
20	**41**	**44**	**47**	**50**	**54**	**57**	**61**	**65**	**68**	**72**
30	**76**	80	85	87	90	94	99	104	109	114
40	120	124	129	134	139	144	150	156	162	168
50	**175**	176	178	181	185	188	192	195	199	203
60	207	212	216	221	226	231	235	240	245	250
70	255	260	265	270	275	280	285	291	296	301
80	306	311	317	323	329	334	340	346	352	357
90	363	368	374	379	385	391	398	404	410	416
100	422	428	434	440	446	452	458	464	470	476
110	483	489	495	501	508	514	520	526	532	538
120	544	551	558	565	571	578	584	590	596	603
130	610	617	623	630	637	644	651	658	665	672
140	679	686	693	700	707	714	721	728	735	742
150	749	756	763	770	777	784	791	798	805	812
160	819	826	834	841	849	856	863	871	878	886
170	893	901	909	917	925	933	941	948	956	963
180	971	979	986	994	1001	1009	1017	1025	1033	1041
190	1049	1057	1065	1073	1081	1089	1097	1105	1113	1121
200	1129									

Table 4.1: Known lower bounds on $ex(n;4)$, from tables printed in [66]. Exact values, when known, are listed in bold font.

(iii) $f_4(2q^2+q-h) \geq q^2(q+1)+(q+1)f_4(q)-h(q+1)$*, for* $q \geq 13$ *and* $h = 1, 2, \ldots q^2$.

Using our GAP algorithm we obtained further improvements on many of the lower bounds on $ex(n;4)$ shown in Tables 4.1 and 4.2. Our improvements are shown in Table 4.3. The graphs that we used as seed graphs for our algorithm are the $(k,5)$-cages, for $3 \leq k \leq 7$ and the $(k,5)$-graphs, for $8 \leq k \leq 12$, (see Table 3.1). The $(k,5)$-graphs produce the lower bounds on $ex(n;4)$ shown in Table 4.3, for $n = 80, 90, 126, 156$ and 203.

A compilation of the current best known upper and lower bounds on $ex(n;4)$, for $n \leq 200$ is given in Appendix B Table B.1. Note that $ex_u(n;4) - ex_l(n;4) \leq 7$, for $n \leq 50$ and we suspect that some of these lower bounds may be extremal. For this reason, we were particularly pleased to be able to improve the lower bounds on $ex(35,4)$ and $ex(45;4)$, although we only increased the lower bound by one. On the other hand, although we were able to increase the lower bounds on $ex(100;4)$ and $ex(200;4)$ by 21 and 44 respectively, the difference between the

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
100					448	456				
110										
120								592	600	608
130	616	624	632	640	648	657	666	669	672	675
140										
150							792	801	810	819
160	828	838	847	846	865	874	883	892	901	910
170	920	930	932	935	938	941	944			

Table 4.2: Improved lower bounds on $ex(n; 4)$ obtained by the constructions in Theorem 4.4.

current best known upper and lower bounds is still large, namely, $443 \leq ex(100; 4) \leq 497$ and $1184 \leq ex(200; 4) \leq 1410$.

The graph that provides the improved lower bound $ex_l(35, 4) = 95$ was obtained by running our Prune algorithm using the $(6, 5)$-cage as input. Similarly, using the Hoffman-Singleton graph as a seed graph we improved the lower bound $ex_l(45; 4) = 145$. It is interesting to note that our Prune algorithm, when given the Hoffman-Singleton graph as input produces the $(6, 5)$ and $(5, 5)$-cages.

4.5 New Families of Extremal Graphs

In this section we present some constructions of graphs that form infinite families of extremal graphs. In order to prove that these graphs are extremal we require the recent results by Abajo and Diánez [2] given in Theorems 4.5 and 4.6. Applying these theorems determines the exact value of the extremal number, for $t \geq 4$ and $n \leq \lfloor (16t - 15)/5 \rfloor$.

Abajo and Diánez [2] established the extremal number, for $t \geq 4$ and $n \leq \lfloor (16t - 15)/5 \rfloor$, as follows. Let $k \geq 0$ be an integer, then, for each $t \geq 2\, log_2\,(k + 2)$ there exists n such that every extremal graph G with $m - n = k$ has minimum degree at most 2, and is obtained by adding vertices of degree 1 and/or subdividing a graph or a multigraph H with $\delta(H) \geq 3$ and $|E(H)| - |V(H)| = k$.

Theorem 4.5 *[2] Let $n \geq 4$ be an integer. Then*

(i) $v_0(t) = t + 1$

(ii) $v_1(t) = \lfloor 3t/2 \rfloor + 1$

(iii) $v_2(t) = 2t$

(iv) $v_3(t) = \begin{cases} \lceil 9t/4 \rceil & \textit{if } t \textit{ is even;} \\ \lfloor 9t/4 \rfloor & \textit{if } t \textit{ is odd.} \end{cases}$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
30						95				
40						145				
50										
60										
70				271	278	285	291	298	305	312
80	320	322	327	334	341	348	355	362	369	376
90	384	392	399	407	415	423	432	436	438	440
100	443	445	447	450	452	458	465	472	480	488
110	496	504	512	520	528	536	544	552	560	568
120	576	585	593	602	611	620	630	634	638	641
130	644	647	650	653	657	666	674	683	692	700
140	709	717	726	735	744	753	762	771	780	789
150	798	808	817	827	837	847	858	862	865	868
160	871	873	875	878	880	883	886			
170							949	958	968	977
180	986	995	1004	1013	1022	1032	1042	1052	1062	1072
190	1082	1092	1102	1112	1122	1132	1142	1152	1163	1173
200	1184	1195	1206	1218	1223	1227	1231	1236	1240	1245

Table 4.3: $\diamondsuit$ Improved lower bounds on $ex(n;4)$ produced by application of our GAP algorithm.

(v) $v_4(t) = \begin{cases} \lceil (8t-2)/3 \rceil & \textit{if } t \neq 4 \textit{ is even;} \\ \lfloor (8t-2)/3 \rfloor & \textit{if } t \textit{ is odd.} \end{cases}$

(vi) $v_5(t) = \begin{cases} 3t-2 & \textit{if } t \neq 6; \\ 3t-1 & \textit{if } t = 6. \end{cases}$

(vii) $v_6(4) = 12, v_6(5) = 14, v_6(6) = 19, v_6(7) = 21,$

$$\lceil (16t-14)/5 \rceil \leq v_6(t) \leq \begin{cases} \lceil (10t-5)/3 \rceil & \textit{if } t \neq 8 \textit{ is even;} \\ \lfloor (10t-6)/3 \rfloor & \textit{if } t \geq 9 \textit{ is odd.} \end{cases}$$

The values of $v_k(t)$ given in Theorem 4.5 are used by Theorem 4.6 to determine $ex(n;t)$, for $n \leq \lfloor (16t-15)/5 \rfloor$.

Theorem 4.6 *[2] Let $t \geq 4$ and $k \geq 0$ be integers. If $v_k(t) \leq n < v_{k+1}(t)$, then $ex(n;t) = n+k$.*

It was suggested by Abajo and Diánez [2] that part of the study of extremal graphs can be restricted to determining graphs or multigraphs whose subdivision produces extremal graphs. Subsequently, Balbuena, Cera, Diánez and García-Vázquez [14] found that subdividing the complete bipartite graphs $K_{3,3}$ produces an extremal graph $G \in EX(15;7)$. In the following theorems, we use current known extremal numbers, which can be calculated by application of Theorem 4.5 and Theorem 4.6, to prove that subdivision of the Petersen graph; the complete graphs K_2, K_3 and K_4; and the complete bipartite graphs $K_{2,3}$, $K_{3,3}$ and $K_{3,4}$; form infinite families of extremal graphs. Furthermore, we prove that subdivision of the Heawood and Tutte-Coxeter graphs produces extremal graphs, thereby, establishing two infinite series of previously unknown extremal numbers.

Recall from Section 2.3, given a graph G of order n and size m and girth g, the subdivided graph s_iG has order $n + mi$ size $m(1 + i)$ and girth $g(1 + i)$.

$\diamondsuit$ **Theorem 4.7** *The subdivided Petersen graphs, s_iP, are an infinite family of extremal graphs $s_iP \in EX(10 + 15i; 4 + 5i)$ and $ex(10 + 15i; 4 + 5i) = 15 + 15i$, for $i \geq 0$.*

Proof. Garnick, Kwong and Lazebnik [66] showed that the Petersen graph is the unique extremal graph, for $n = 10$ and $t = 4$, that is, $EX(10; 4) = \{P\}$ and $ex(10; 4) = 15$. The subdivided Petersen graph s_iP has order $n = 10 + 15i$, girth $g = 5 + 5i$ and size $m = 15 + 15i$.

Let $t = g - 1 = 4 + 5i$. Then applying Theorem 4.5 (vi), we have $v_5(4 + 5i) = 3(4 + 5i) - 2 = 10 + 15i = n$. Then, by Theorem 4.6, we know that $ex(10 + 15i; 4 + 5i) = n + 5 = 15 + 15i$. Therefore, $s_iP \in EX(10 + 15i; 4 + 5i)$. ■

$\diamondsuit$ **Theorem 4.8** *The subdivided complete graphs, s_iK_2, s_iK_3 and s_iK_4, for $i \geq 1$ form infinite families of extremal graphs.*

(i) $s_iK_2 \in EX(2 + i; t)$ *and* $ex(2 + i; t) = 2 + i - 1$, *for* $3 \leq t \leq n$.
(ii) $s_iK_3 \in EX(3 + 3i; t)$ *and* $ex(3 + 3i; t) = 3 + 3i$, *for* $t + 1 \leq n \leq \lfloor 3t/2 \rfloor$ *and* $i \geq 1$.
(iii) $s_iK_4 \in EX(4 + 6i; 2 + 3i)$ *and* $ex(4 + 6i; 2 + 3i) = 6 + 6i$, *for* $i \geq 1$.

Proof. (i) For $i = 0$, the complete graph K_2 is the path on two vertices and is known to be extremal $EX(2; t) = \{K_2\}$, for $3 \leq t \leq n$ and $ex(2; t) = 1$. Subdividing K_2 by i adds i edges and i vertices and the graph retains the property of being acyclic. Therefore, $s_iK_2 = P_{2+i} \in EX(2 + i; t)$ and $ex(2 + i; t) = 2 + i - 1$, for $3 \leq t \leq n$.

(ii) For $i = 1$, the subdivided complete graph $s_1K_3 = C_6$ which is the extremal graph $C_6 \in EX(6; t)$, for $t + 1 \leq n$. Each subdivision of the graph adds 3 edges and 3 vertices and increases the girth by three. Thus creating the cycles $s_iK_3 = C_{3+3i} \in EX(3 + 3i; t)$ which are known to be extremal, for $t + 1 \leq n \leq \lfloor 3t/2 \rfloor$.

(iii) The graph K_4 has 4 vertices, 6 edges and girth 3. Subdividing K_4 by i adds $6i$ vertices, $6i$ edges and increases the girth by $3i$. Thus $|V(s_iK_4)| = 4 + 6i$, $|E(s_iK_4)| = 6 + 6i$ and the girth is $g(s_iK_4) = 3 + 3i$. Since $t \leq g - 1 = 3 + 3i - 1 = 2 + 3i$. We need to show that $ex(4 + 6i; 2 + 3i) = 6 + 6i$. Notice that $n = 4 + 6i = 2(2 + 3i) = 2t$. Applying Theorem 4.5 (ii) and Theorem 4.6 if $n = v_2(t) = 2t$, then $ex(n; t) = n + 2$ as required. ■

It is easily determined that the subdivided complete graphs s_iK_n, for $n \geq 5$ are not extremal, for example, s_1K_5 has order 15, size 20 and girth 6 while $ex(15; 5) = 22$.

$\diamondsuit$ **Theorem 4.9** *The subdivided complete bipartite graphs, $s_iK_{2,3}$, $s_iK_{3,3}$ and $s_iK_{3,4}$ form infinite families of extremal graphs and,*

(i) $s_iK_{2,3} \in EX(5+6i; 3+4i)$ *and* $ex(5+6i; 3+4i) = 6+6i$.
(ii) $s_iK_{3,3} \in EX(6+9i; 3+4i)$ *and* $ex(6+9i; 3+4i) = 9+9i$.
(iii) $s_iK_{3,4} \in EX(7+12i; 3+4i)$ *and* $ex(7+12i; 3+4i) = 12+12i$.

Proof. (i) For $i = 0$, the fact that $K_{2,3}$ is extremal is a direct result of Mantel's theorem and $K_{2,3} \in EX(5, 3)$. Furthermore, $|V(K_{2,3})| = 5$, $|E(K_{2,3})| = 6$ and $g(K_{2,3}) = 4$. Subdividing $K_{2,3}$ by i adds $6i$ vertices and $6i$ edges and increases the girth by $4i$. Applying Theorem 4.5 (i) and Theorem 4.6 if $n = v_1(t) = \lfloor 3(3+4i)/2 \rfloor + 1 = 4+6i+1 = 5+6i$, then $ex(n; t) = n+1$, therefore $ex(5+6i; 3+4i) = 6+6i$ as required.

(ii) For $i = 0$, the fact that $K_{3,3}$ is extremal is a direct result of Mantel's theorem and $K_{3,3} \in EX(6, 3)$. Furthermore, $|V(K_{3,3})| = 6$, $|E(K_{3,3})| = 9$ and $g(K_{3,3}) = 4$. Subdividing $K_{3,3}$ by i adds $9i$ edges and $9i$ vertices and increases the girth by $4i$. Applying Theorem 4.5 (vi), for $t \neq 6$ and Theorem 4.6 if $n = 6+9i \leq \lfloor 9t/4 \rfloor = \lfloor 9(3+4i)/4 \rfloor = 27/4+9i$, then $ex(n; t) = n+3$. Therefore, $ex(6+9i; 3+4i) = 9+9i$ as required.

(iii) For $i = 0$, the fact that $K_{3,4}$ is extremal is a direct result of Mantel's theorem and $K_{3,4} \in EX(7, 3)$. Furthermore, $|V(K_{3,4})| = 7$, $|E(K_{3,4})| = 12$ and $g(K_{3,4}) = 4$. Subdividing $K_{3,4}$ by i adds $12i$ edges and $12i$ vertices and increases the girth by $4i$. Applying Theorem 4.5 (vi), for $t \neq 6$ and Theorem 4.6 if $n = 7+12i \leq 3t-2 = 3(3+4i)-2 = 7+12i$, then $ex(n; t) = n+5$. Therefore, $ex(7+12i; 3+4i) = 12+12i$ as required. ■

The subdivided complete bipartite graph $s_1K_{4,4}$ is extremal, that is, $s_1K_{4,4} \in EX(24; 7)$ and $ex(24; 7) = 32$. The graph $s_2K_{4,4}$ gives the lower bound $ex_l(40; 11) = 48$ and the upper bound due to the Moore bound for irregular graphs is $ex_u(40; 11) = 49$ so there is a good chance that this graph is extremal. However, it is not yet known if the graphs $s_iK_{4,4}$, for $i \neq 1$ are extremal or not.

The above theorems make use of known extremal numbers to prove that the subdivided graphs are extremal. In the following theorems we show that the subdivided Heawood graph and the Tutte-Coxeter graph are infinite families of extremal graphs. Correspondingly, we establish two infinite series of previously unknown extremal numbers.

$\diamondsuit$ **Theorem 4.10** *The subdivided Heawood graphs, s_iH, are an infinite family of extremal graphs $s_iH \in EX(14+21i; 5+6i)$ and $ex(14+21i; 5+6i) = 21+21i$.*

Proof. For $i = 0$, the Heawood graph is a Moore cage and therefore extremal and the corresponding extremal number is $ex(14; 5) = 21$. Subdividing the Heawood graph by $i \geq 1$ adds $21i$ edges and $21i$ vertices and increases the girth by $6i$. Lower bounds on the extremal number due to this construction are $ex(14+21i; 5+6i) \geq 21+21i$.

The girth of s_iH is $g = 6i+6$ and the average degree is $\overline{d} = 2(21+21i)/(14+21i) = (6+6i)/(2+3i)$. Substituting these values into the Moore bound for irregular graphs by Alon, Hoory and Linial we have

$$2\sum_{k=0}^{3i+2}((6+6i)/(2+3i)-1)^k < 14+21i$$

Assume $ex(14+21i; 5+6i) > 21+21i$. Then,

$$2\sum_{k=0}^{3i+2}(2(22+21i)/(14+21i)-1)^k < 14+21i$$

Which is impossible. Therefore, $ex(14+21i; 5+6i) = 21+21i$ as required. ■

$\diamondsuit$ **Theorem 4.11** *The subdivided Tutte-Coxeter graphs, s_iTC, are an infinite family of extremal graphs $s_iTC \in EX(30+45i; 7+8i)$ and $ex(30+45i; 7+8i) = 45+45i$.*

Proof. For $i = 0$, the Tutte-Coxeter graph, or (3,8)-cage, is a Moore cage and therefore extremal and the corresponding extremal number is $ex(30; 7) = 45$. Subdividing the Tutte-Coxeter graph by $i \geq 1$ adds $45i$ edges and $45i$ vertices and increases the girth by $8i$. Lower bounds on the extremal number due to this construction are $ex(30+45i; 7+8i) \geq 45+45i$.

The girth of s_iTC is $g = 7+8i$ and the average degree is $\overline{d} = 2(45+45i)/(30+45i) = (6+6i)/(2+3i)$. Substituting these values into the Moore bound for irregular graphs by Alon, Hoory and Linial we have

$$2\sum_{k=0}^{4i+3}((6+6i)/(2+3i)-1)^k < 30+45i$$

Assume $ex(30+45i;7+8i) > 45+45i$. Then,

$$2\sum_{k=0}^{4i+3}(2(46+45i)/(30+45i)-1)^k < 45+45i$$

Which is impossible. Therefore, $ex(30+45i;7+8i) = 45+45i$ as required. ■

4.6 $ex(n;t)$, for $t \in \{5,6,7\}$

Abajo and Diánez [2] determined the exact values of: $ex(n;5)$, for $n \leq 13$; $ex(n;6)$, for $n \leq 16$; and $ex(n;7)$, for $n \leq 19$. Perhaps motivated by this progress, Tang, Lin, Balbuena and Miller [118] used hybrid simulated annealing and genetic algorithm to produce constructive lower bounds on the function $ex(n;t)$, for $t \in \{5,6,7\}$ and $n \leq 39$. Subsequently, Abajo and Diánez [3], confirmed the lower bounds generated by Tang, Lin, Balbuena and Miller [118] to be the exact values of $ex(n \leq 27;5)$, $ex(n \leq 28;6)$ and $ex(n \leq 34;7)$. Furthermore, the same authors developed a greedy algorithm that makes use of a good graph and adds paths and subdivides arbitrary edges to create new graphs. Using this algorithm they improved the lower bounds on: $ex(n;5)$, for $28 \leq n \leq 62$; $ex(n;6)$, for $29 \leq n \leq 49$; and $ex(n;7)$, for $35 \leq n \leq 80$. Moreover, authors of [3] proved that a number of the lower bounds generated by their algorithm were, in fact, extremal. In particular: $ex(n;5)$, for $28 \leq n \leq 42$ and $n = 62$; and $ex(n;7)$, for $n = 35, 36, 160$. Subsequently, Tang, Lin, Balbuena and Miller [119] proved that $ex(29;6) = 45$. Delorme, Flandrin, Lin, Miller and Ryan [44] provided sketches of proofs, for $ex(29;6) = 45$, $ex(30;6) = 47$ and $ex(31;6) = 49$. More recently, Abajo, Balbuena and Diánez [1] provided further improved lower bounds on $ex(n;6)$, for $n \leq 300$.

The above mentioned results are summarised in Tables 4.4, 4.6 and 4.8.

Using knowledge of the extremal number, $ex(24;6)$, Abajo and Diánez [3] were able to prove that the McGee graph is the unique extremal graph, for $EX(24;6)$. Using the same line of reasoning we show that the Heawood and Tutte-Coxeter cages are unique extremal graphs, for $EX(14;5)$ and $EX(30;7)$, respectively.

♢ **Corollary 4.1** *The Heawood graph is the unique extremal graph, for* $EX(14;5)$.

Proof. From Table 4.4 we know that $ex(14;5) = 21$. Therefore, the average degree of the graph $G \in EX(14;5)$ is 3. Using Observation 4.4 and the value of $ex(13;5)$ from Table 4.4 we know that $ex(14;5) - ex(13;5) = 21 - 18 = 3 \leq \delta$. Therefore, G is 3-regular. The Heawood graph is the (3,6)-cage which is the point-line incidence graph of the generalised triangle as shown in Figure 2.7. This graph is known to be unique [56]. ■

◇ **Corollary 4.2** *The Tutte-Coxeter cage is the unique extremal graph, for $EX(30;7)$.*

Proof. From Table 4.8 we know that $ex(30;7) = 45$. Therefore, the average degree a graph $G \in EX(30;7)$ is 3. Using Observation 4.4 and the value of $ex(29;7)$ from Table 4.8 we know that $ex(30;7) - ex(29;7) = 45 - 42 \leq 3$. Therefore, G is 3-regular. The Tutte-Coxeter cage is the (3,8)-cage which is the point-line incidence graph of the generalised quadrangle of order 2. This graph is known to be unique [56]. ■

4.6.1 $ex(n;5)$

Our GAP algorithm provided no improvements on the current best known upper and lower bounds on $ex(n;5)$, for $n \leq 62$ that are summarised in Table 4.4. We are not particularly disheartened by this result since there are only 19 values of n in this range for which the extremal number is not yet known. Furthermore, $ex_u(n;5) - ex_l(n;5) \leq 7$, for $n \leq 62$ which indicates that these bounds are already quite tight. Nevertheless, we were able to generate new lower bounds on $ex(n;5)$, for $63 \leq n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**7**	**8**	**9**	**10**
10	**12**	**14**	**16**	**18**	**21**	**22**	**24**	**26**	**29**	**31**
20	**34**	**36**	**39**	**42**	**45**	**48**	**52**	**53**	**56**	**58**
30	**61**	**64**	**67**	**70**	**74**	**77**	**81**	**84**	**88**	**92**
40	**96**	**100**	**105**	106-108	108-112	110-116	114-119	118-123	122-127	125-131
50	130-135	134-139	138-143	142-147	147-151	151-155	156-160	160-164	165-168	170-172
60	175-177	180-181	**186**							

Table 4.4: Known upper and lower bounds on $ex(n;5)$ as given in [1]. Exact values, when known, are listed in bold font.

The graphs that were used as input for our GAP algorithm for the new lower bounds on $ex(n;5)$ were the $(k,6)$-cages, for $6 \leq k \leq 10$. With the exception of the (7,6)-cage, these graphs are the incidence graphs of the generalised triangles which are Moore cages and, consequently, extremal. Thus, $ex(114;5) = 456$, $ex(146;5) = 657$ and $ex(182;5) = 910$. The (7,6)-cage gives the new lower bound $ex(90;5) \geq 315$. Furthermore, we used Brown's [8] construction for Hamiltonian regular graphs of girth six, for $k = 10$, which produces a (10,6)-graph of order 200 which attains the current best known lower bound on $ex(200;6)$. We used knowledge of the girth and degree regularity of Brown's graph to manually calculate new lower bounds on

$ex(n;5)$, for $192 \le n \le 200$. The new best known lower bounds on $ex(n;5)$ generated by our GAP algorithm using the above mentioned seed graphs are given in Table 4.5.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
60				187	189	191	193	195	199	204
70	208	212	217	222	227	232	237	242	247	252
80	257	262	268	273	279	284	290	296	302	308
90	315	318	322	325	329	334	339	344	350	356
100	362	368	375	381	388	394	401	407	414	420
110	427	434	441	448	**456**	457	459	461	463	465
120	467	469	475	482	489	496	504	511	519	526
130	534	541	549	556	564	571	579	586	594	601
140	609	616	624	632	640	648	**657**	658	660	662
150	664	666	668	670	672	679	687	695	703	711
160	720	728	737	745	754	762	771	779	788	796
170	805	813	822	830	839	847	856	864	873	882
180	891	900	**910**	911	913	915	917	919	921	923
190	925	927	929	937	946	954	963	972	981	990
200	1000									

Table 4.5: ♢ New lower bounds on $ex(n;5)$ produced by application of our GAP algorithm.

A compilation of the current best known upper and lower bounds on $ex(n;5)$, for $n \le 200$ is given in Appendix B Table B.2. Note that the difference between the current best known upper and lower bounds on $ex(n;5)$ is quite small in comparison to those of $ex(n;4)$, for example, $ex_u(100;4) - ex_l(100;4) = 54$ compared to $ex_u(100;5) - ex_l(100;5) = 14$ and $ex_u(200;4) - ex_l(200;4) = 226$ compared to $ex_u(200;5) - ex_l(200;5) = 46$. Furthermore, when $ex(n;5)$ is known then $ex_u(n+1;5) - ex_l(n+1;5) \le 6$ and $ex_u(n-1;5) - ex_l(n-1;5) \le 2$.

4.6.2 $ex(n;6)$

Recently, Abajo, Balbuena and Diánez [1] constructed some infinite families of graphs that produce the current best known lower bounds on $ex(n;6)$. The bounds produced by these constructions are given in Theorem 4.12. These values include the extremal numbers $ex(24;6) = 36$, $ex(26;6) = 39$ and the current best known lower bounds $ex_l(72;6) = 147$, $ex_l(75;6) = 152$, $ex_l(160;6) = 408$ and $ex_l(162;6) = 415$. Furthermore, Abajo, Balbuena and Diánez generated new lower bounds on $ex(n;6)$, for $n \le 300$. In Table 4.6 we have collated these new lower results with those previously mentioned.

Theorem 4.12 *[1] Let $f_6(n) = ex(n;6)$ and $q \ge 3$ be a prime power. Then*

(i) $f_6(2q^3 + 2q^2 + q) \ge q^2(q+1)^2 + (q+1)f_6(q)$.

(ii) $f_6(2(q^3+q^2)) \geq q^4+2q^3+2f_6(q)$.

(iii) $f_6(2(q^3+q^2)-h) \geq q^4+2q^3+2f_6(q)-h(q+1)$, *for all* $h=1,2,\ldots q^3$.

In Table 4.7 we list the improved lower bounds on $ex(n;6)$ produced by our GAP algorithm. We used the (4,7)-cage and (5,7)-graph on 152 vertices, which provides the current best known upper bound on $n(5,7)$, as input graphs.

In the following theorems we determine the previously unknown extremal number $ex(n;6)$, for $n=30,31,32$. In order to do this we adopt the following strategy.

1. Determine the current best known lower bound on $ex_l(n;t)$ from Tables 4.6 and 4.7
2. Assume the existence of a graph $G \in EX(n;t)$ of order $ex_l(n;t)+1$.
3. Determine a number of necessary structural properties of G that we then use to prove that G does not exist.
4. Apply Abajo and Diánez's [3] observation that proving the non-existence of a graph of order $ex_l(n;t)+1$ and girth $g>t$ is sufficient to prove $ex(n;t) \leq ex_l(n;t)$.
5. Conclude that $ex_l(n;t)=ex(n;t)$.

We use the notation introduced in Section 4.1, namely, $T_{\Delta,\delta,t}$ denotes the tree having height $\lfloor (t+1)/2 \rfloor$ and root r such that $deg(r)=\Delta$, the leaves have degree 1, and every other vertex $v \neq r$ has $deg(v)=\delta$. Since $g \geq t+1$, every $G \in EX(n;t)$ must contain $T_{\Delta,\delta,t}$ as a subgraph. Furthermore,

$$|V(G)| \geq |V(T_{\Delta,\delta,t})| = \begin{cases} 1+\sum_{i=1}^{t/2} \Delta(\delta-1)^{i-1} & \text{for even } t, \\ 1+\sum_{i=1}^{(t-1)/2} \Delta(\delta-1)^{i-1}+(\delta-1)^{(t-1)/2} & \text{for odd } t. \end{cases}$$

We use X to denote the set of vertices $X=V(G-T_{\Delta,\delta,t})$. We use the notation $T_{\Delta,\delta,t}(\mathcal{F})$ to represent the tree $T_{\Delta,\delta,t}$ with additional forbidden subgraph constraints, that is, if $F \subset \mathcal{F}$ then $F \not\subseteq T_{\Delta,\delta,t}$. We use the term *n-star* and notation S_n to mean the complete bipartite graph $K_{1,n-1}$. The path on k vertices such that all k vertices in the path have degree j in G is denoted P_k^j.

In the following two theorems we provide full proofs, for $ex(30;6)=47$ and $ex(31;6)=49$. Sketches of proofs for these values were given by Delorme, Flandrin, Lin, Miller and Ryan [44].

♢ **Lemma 4.1** *Assume* $ex(30;6)=48$ *and let* $G \in EX(30;6)$. *Then*

(i) The minimum degree is 3 and the maximum degree is 4.

(ii) There are six vertices of degree 4 and twenty-four vertices of degree 3 in G.

(iii) Let r *be a vertex of degree 4. Then there is at most one vertex of degree 4 at distance 2 from* r. *All other vertices of degree 4 are at distance 3 or more from* r.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**7**	**8**	**9**
10	**11**	**12**	**14**	**15**	**17**	**18**	**20**	**22**	**23**	**25**
20	**27**	**29**	**31**	**33**	**36**	**37**	**39**	**41**	**43**	**45**
30	47-49	48-52	50-54	52-56	55-58	57-61	59-63	61-65	62-68	64-70
40	67-73	69-75	71-77	73-80	75-82	77-85	80-87	82-90	84-92	87-95
50	89	91	93	95	97	99	102	105	107	109
60	112	115	117	120	123	125	128	131	134	137
70	140	143	147	148	150	152	154	156	158	160
80	162	164	166	168	171	174	176	178	180	182
90	184	186	188	190	193	196	198	201	204	206
100	209	212	214	217	220	223	225	228	231	234
110	237	240	243	246	249	252	255	258	261	265
120	268	271	274	278	281	284	287	291	294	297
130	301	305	308	311	314	317	320	323	327	330
140	333	337	340	343	347	351	354	357	361	365
150	368	372	376	369	383	387	391	395	399	403
160	408	409	411	413	415	417	419	421	423	425
170	427	429	431	433	435	437	439	441	443	445
180	447	449	451	453	455	458	460	462	464	467
190	469	471	473	475	478	482	485	489	492	496
200	499	503	506	510	513	517	520	524	527	531
210	535	539	543	546	550	554	558	561	565	569
220	573	577	581	585	588	592	596	600	604	608
230	612	616	620	624	628	632	636	640	644	648
240	652	656	660	664	668	672	676	680	684	688
250	693	697	701	705	709	714	718	722	726	730
260	734	738	742	746	751	755	759	763	768	772
270	776	781	785	789	794	798	802	807	811	816
280	820	825	829	833	838	843	847	851	856	861
290	865	870	875	879	884	889	894	899	904	909
300	915									

Table 4.6: Known upper and lower bounds on $ex(n;6)$, for $n \leq 16$ from [2], for $17 \leq n \leq 28$ from [3] and $ex(29;6) = 45$ [119]. Exact values, when known, are listed in bold font.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
30		49	51	53		58			63	65
40						78	81	83	86	88
50	91	93	95	97	100	102	104	106	108	110
60			118	121	124	127	130	134	136	138
140				344	348	352	355	359	363	367
150	371	375	380	382	384					

Table 4.7: ♢ Improved lower bounds on $ex(n; 6)$ produced by application of our GAP algorithm.

Proof. (i) We know $ex(29; 6) = 45$ from Table 4.6. Applying Inequality 4.5,

$$ex(30; 6) - ex(29; 6) = 48 - 45 = 3 \leq \delta \leq 3 = \lfloor (48 \times 2)/30 \rfloor \leq \lceil (48 \times 2)/30 \rceil = 4 \leq \Delta$$

gives $\delta = 3$ and $\Delta \geq 4$. Therefore, G contains the tree $T_{\Delta,3,6}$ as a subgraph and

$$|V(G)| = 30 \geq |V(T_{\Delta,3,6})| = 1 + |N(r)| + |N_2(r)| + |N_3(r)| = 1 + \Delta + 2\Delta + 4\Delta = 1 + 7\Delta.$$

Therefore, $\Delta = 4$.

(ii) Let x be the number of vertices of degree 3 in G and y the number of vertices of degree 4. Then $x + y = 30$ and $3x + 4y = 48 \times 2 = 96$. Solving these equations determines that there are twenty-four vertices of degree 3 and six vertices of degree 4, thus the degree sequence is $\mathcal{D}= (4^6, 3^{24})$.

(iii) Since $\Delta = 4$ and $\delta = 3$, the tree $T_{4,3,6}$ is a subgraph of G. Therefore, $|V(G)| = 30 \geq |V(T_{4,3,6})| = 29$ and $X = V(G - T_{4,3,6}) = \{x\}$. Assume $x \in N(r)$. Then $deg(r) = 5$ which is impossible because $\Delta = 4$. Assume $x \in N_2(r)$. Then $|V(G)| = 30 \geq \sum_{i=0}^{3} |N_i(r)| = 1 + 4 + 9 + 18 = 32$. Assume $x \in N_3(r)$ (see Figure 4.2). Then there is one vertex u of degree 4 in $N_2(r)$ and the other four vertices of degree 4 must be in $N_3(r)$. Furthermore if $x \in N_4(r)$, then all vertices of degree 4 are at distance 3 or more from r. ■

♢ **Theorem 4.13** *Let $G \in EX(30; 6)$. Then $ex(30; 6) = 47$.*

Proof. The new lower bound on $ex(30; 6)$ due to our GAP algorithm is $ex_l(30; 6) = 47$ (see Table 4.7). Assume $ex(30; 6) = 48$. Then by Lemma 4.1 the degree sequence is $\mathcal{D}= (4^6, 3^{24})$ and the tree $T_{4,3,6}$ is a subgraph of G. Consider the six vertices that have degree 4 in G as S_5 structures, as shown in Figure 4.3. From Lemma 4.1 (iii) there is at most one vertex, u, such that, $deg(u) = 4$ and $d(u, r) = 2$.

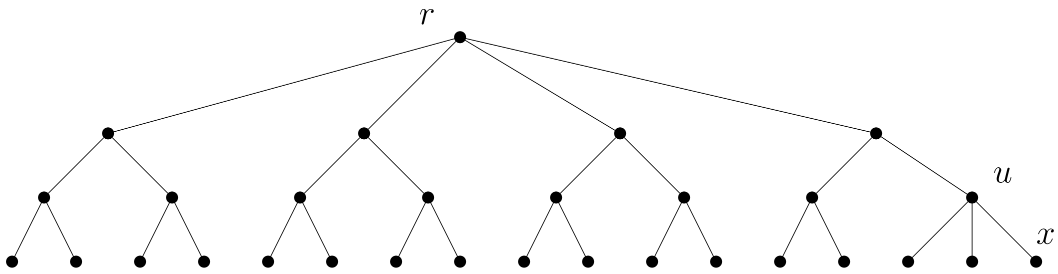


Figure 4.2: $T_{4,3,6}$ with one additional vertex of degree 4.

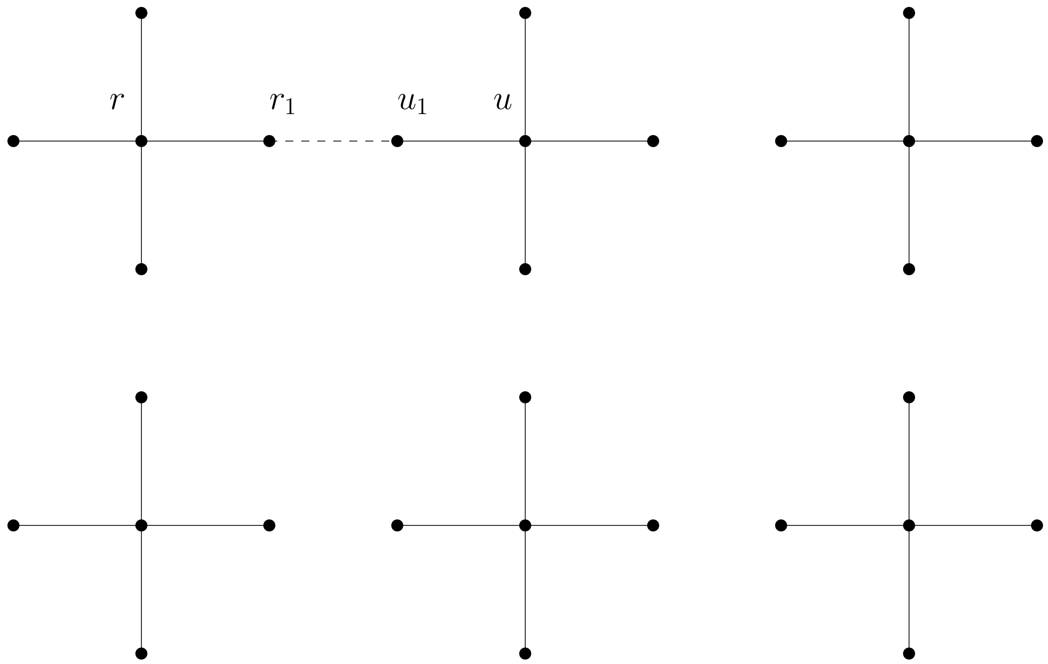


Figure 4.3: Six S_5 structures.

Assume that the distance between any two vertices of degree 4 in G is at least 3. Then the six 5-stars are distinct and unique and there are exactly $6 \times 5 = 30$ vertices and $6 \times 4 = 24$ edges in these structures. There is at most one edge between each pair of 5-stars, otherwise a cycle of length 6 is formed, that is, 5+4+3+2+1=15 edges between the six 5-stars. Let $\mathcal{H}$ denote the graph that is composed of six 5-stars with an edge between each pair of 5-stars. Then $|V(\mathcal{H})| = 30$ and$|E(\mathcal{H})| = 24 + 15 = 39$.

Assume that there is one vertex u of degree 4, such that, $d(u, r) = 2$ in G. This is equivalent to beginning with the graph $\mathcal{H}$ and contracting an edge between one pair of 5-stars as illustrated by the dotted line, labeled r_1u_1 in Figure 4.3. Let $\mathcal{H}'$ be the graph obtained by contracting an edge in $\mathcal{H}$. Then $|V(\mathcal{H}')| = |V(\mathcal{H}' - \{x\})| = 30 - 1 = 29$ and $|E(\mathcal{H}')| = |E(\mathcal{H}' - \{r_1u_1\})| = 39 - 1 = 38$ and there is one vertex $x \notin \mathcal{H}'$ such that $deg(x) = 3$. Then, $|V(\mathcal{H}' \cup \{x\})| = 30$ and $|E(\mathcal{H}' \cup \{x\})| = 38 + 3 = 41$

Therefore, $ex(30; 6) = 47$. ■

We can use the results of Theorem 4.13, that is, $ex(30; 6) = 47$ to lower the upper bounds on $ex(31; 6) = 51$ as follows. We know that $49 \leq ex(31; 6) \leq 51$ (see Table 4.6). Assume $ex(31; 6) = 51$.Then applying Inequality 4.4, $ex(31; 6) - \delta \leq ex(30; 6)$, determines that $\delta \geq 51-47 = 4$. Applying Inequality 4.3, $\delta \leq 3 = \lfloor (51 \times 2)/31 \rfloor$ which is a contradiction. Therefore, $49 \leq ex(31; 6) \leq 50$.

Although this is a useful technique, we prefer, where possible, to generate lower bounds that are equal to the extremal number and use the observation of Abajo and Diánez [3], namely, in order to prove $ex(n; t) \leq m$ it is sufficient to prove the non-existence of a graph of order $m + 1$ and girth $g > t$. Therefore, in the following lemma we assume that $ex(31; 6) = 50$.

♢ **Lemma 4.2** *Assume $ex(31; 6) = 50$ and let $G \in EX(31; 6)$ then*

(i) The minimum degree is 3 and the maximum degree is 4.

(ii) There are seven vertices of degree 4 and twenty-four vertices of degree 3 in G.

(iii) A vertex of degree 4 has no neighbours of degree 4.

(iv) Given a vertex r in G, such that $deg(r) = 4$, there are at most two vertices of degree 4 at distance 2 from r.

Proof. (i) We know $ex(30; 6) = 47$ from Theorem 4.13. Applying Inequality 4.5,

$$ex(31; 6) - ex(30; 6) = 50 - 47 = 3 \leq \delta \leq 3 = \lfloor (50 \times 2)/31 \rfloor \leq \lceil (50 \times 2)/ \rceil = 4 \leq \Delta$$

Therefore, $\delta = 3$ and $\Delta \geq 4$. Then $T_{\Delta,3,6}$ is a subgraph of G and $|V(G)| = 31 \geq |V(T_{\Delta,3,6})| = 1 + 7\Delta$. Therefore, $\Delta = 4$.

(ii) Let x be the number of vertices of degree 3 in G and y be the number of vertices of degree 4. Then $x + y = 31$ and $3x + 4y = 100$. Solving these equations we obtain the degree sequence $\mathcal{D}= (4^7, 3^{24})$.

(iii) Assume G contains two vertices of degree 4 that are neighbours. Then G has a subgraph which is a tree of height 3, with root vertex r such that $deg(r) = 4$ and another vertex $u \in N(r)$ such that $deg(u) = 4$ and all other vertices in T have degree 3. Then $|V(G)| \geq |V(T)| = 1 + N(r) + N_2(r) + N_3(r) = 1 + 4 + 9 + 18 = 32$. A contradiction. Therefore, a vertex of degree 4 has no neighbours of degree 4.

(*iv*) Assume G contains three vertices of degree 4 at distance 2 from a vertex r, where $deg(r) = 4$. Then

$$\sum_{i=0}^{3} |N_i(r)| = 1 + 4 + 8 + 19 = 32.$$

Impossible. Therefore, there are at most two vertices of degree 4 at distance 2 from a vertex r, such that $deg(r) = 4$. ■

◇ **Theorem 4.14** *Let* $G \in EX(31; 6)$*. Then* $ex(31; 6) = 49$.

Proof. Assume $ex(31; 6) = 50$. Then by Lemma 4.2 (*ii*) the degree sequence of G is $\mathcal{D} = (4^7, 3^{24})$ and there are seven S_5 structures in G. There are 7 x 4 = 28 edges in the seven S_5 structures. There is at most one edge between every pair of S_5 structures, otherwise a cycle of length 6 is formed. Therefore, there are, at most 6+5+4+3+2+1=21 edges between the structures. Let $\mathcal{H}$ denote the graph that is composed of seven 5-stars with an edge between each pair of 5-stars. Then $|V(\mathcal{H})| = 35$ and $|E(\mathcal{H})| = 28 + 21 = 49$.

Since $|V(\mathcal{H})| = 35$ the vertices of the 5-stars are not distinct and unique. Therefore, at least four of the vertices in the S_5 structures must be "shared". Lemma 4.2 (*iii*) asserts that no two vertices of degree 4 are neighbours. Therefore, "sharing" a vertex is equivalent to contracting an edge between a pair of S_5 structures in $\mathcal{H}$.

By Lemma 4.2 (*iv*), given a vertex $r \in G$ such that $deg(r) = 4$, there are at most two vertices of degree 4 at distance 2 from r. If every vertex of degree 4 has two vertices of degree 4 at distance 2, then the vertices of degree 4 lie on a cycle of length 14, where the degrees of the vertices alternate between vertices of degree 3 and vertices of degree 4.

Therefore, there are at least four and at most seven shared vertices. Let $\mathcal{H}^{-i}$ denote the graph that is the graph $\mathcal{H}$ with i of the edges between vertices of degree 3 contracted.

Assume there are 4 shared vertices. Then $|V(\mathcal{H}^{-4})| = 35 - 4 = 31$ and $|E(\mathcal{H}^{-4})| = 49 - 4 = 45$.

Assume there are 5 shared vertices. Then $|V(\mathcal{H}^{-5})| = 35 - 5 = 30$ and $|E(\mathcal{H}^{-5})| = 49 - 5 = 44$ and there is one vertex of degree 3 that is not included in these structures. This vertex can contribute at most 3 extra edges to G. Therefore, there are at most 44+3=47 edges in G.

Assume there are 6 shared vertices. Then $|V(\mathcal{H}^{-6})| = 35 - 6 = 29$ and $|E(\mathcal{H}^{-6})| = 49 - 6 = 43$ and there are two vertices of degree 3 that are not in these structures. These vertices contribute at most 6 additional edges. Therefore, in total there are at most 43+6=49 edges.

Assume there are 7 shared vertices. Then there is a cycle of length 14 in the graph such that every vertex of degree 3 has two neighbours of degree 4 and every vertex of degree 4 has two neighbours of degree 3 on this cycle. In such a graph there are at most $7 \times 4 = 28$ edges in the S_5 structures and $4+3+2+1 = 10$ between the leaves of the S_5 structures and another $3 \times 3 = 9$ edges from the remaining vertices of degree 3, that is, at most $28+10+9 = 47$ edges. Therefore $ex(31;6) = 49$. ■

$\diamondsuit$ **Lemma 4.3** *Assume $ex(32;6) = 52$ and let $G \in EX(32;6)$ then*

(i) The maximum degree is 4 and the minimum degree is 3.

(ii) There are eight vertices of degree 4 and twenty-four vertices of degree 3 in G.

(iii) No two vertices of degree 4 are neighbours.

(iv) A vertex of degree 3 has at most two neighbours having degree 4.

(v) There are eight vertices of degree 3 that have two neighbours having degree 4.

(vi) There are sixteen vertices of degree 3 that have one neighbour having degree 4.

(vii) There are no vertices of degree 3 adjacent only to vertices of degree 3.

Proof. (i) By Theorem 4.14 we know that $ex(31;6) = 49$. Applying Inequality 4.5,

$$ex(32;6) - ex(31;6) = 52 - 49 = 3 \leq \delta \leq 3 = \lfloor (52 \times 2)/32 \rfloor \leq \lceil (52 \times 2)/32 \rceil = 4 \leq \Delta$$

gives $\delta = 3$ and $\Delta \geq 4$. Then $|V(G)| = 32 \geq |V(T_{\Delta,3,6})| = 1 + 7\Delta$. Therefore, $\Delta = 4$.

(ii) Let x be the number of vertices in G having degree 3 and y be the number of vertices of degree 4. Then $x + y = 32$ and $3x + 4y = 104$. Solving the equations we obtain the degree sequence $\mathcal{D}= (4^8, 3^{24})$.

(iii) Assume that vertices $r_1, r_2 \in V(G)$ having $deg(r_1) = deg(r_2) = 4$ are neighbours. Then G contains a subgraph which consists of an edge r_1r_2 and two trees T_{r_1}, T_{r_2} with respective roots r_1, r_2 and $r_1 \notin V(T_{r_2})$, $r_2 \notin V(T_{r_1})$. There are at least

$$|V(T_{r_2})| = |V(T_{r_1})| = 1 + |N(r_1) \cap V(T_{r_1})| + |N_2(r_1) \cap V(T_{r_1})| \geq 1 + 3 + 6 = 10$$

vertices in each tree as shown in Figure 4.4. The vertices in each tree must be distinct and edges exist only between parent and child vertices, otherwise a forbidden cycle is formed. Let X be the set of remaining vertices $X = \{x_1, x_2, \ldots x_{12}\}$, where $x_i \notin V(T_{r_1} \cup T_{r_2})$. Let L be the set of leaf vertices $L = \{l_{11}, l_{12}, \ldots, l_{16}, l_{21}, l_{22}, \ldots, l_{26}\}$, such that, $l_{1k} \in N_2(r_1) \cap V(T_{r_1})$ and $l_{2k} \in N_2(r_2) \cap V(T_{r_2})$ Since $\delta = 3$, each of the 12 leaf vertices have at least two distinct neighbours in the set X. So there are at least 24 edges $l_{j,k}x_i$. However, each vertex x_i can have at most one neighbour in each of the subtrees T_{r_1} and T_{r_2}, otherwise a cycle of length six or less is formed. So there are at most 24 edges $x_il_{j,k}$. Therefore, there are exactly 24 edges between the leaves of T_{r_1} and T_{r_2} and the vertices in X. Therefore, $|V(T_{r_2})| = |V(T_{r_1})|$.

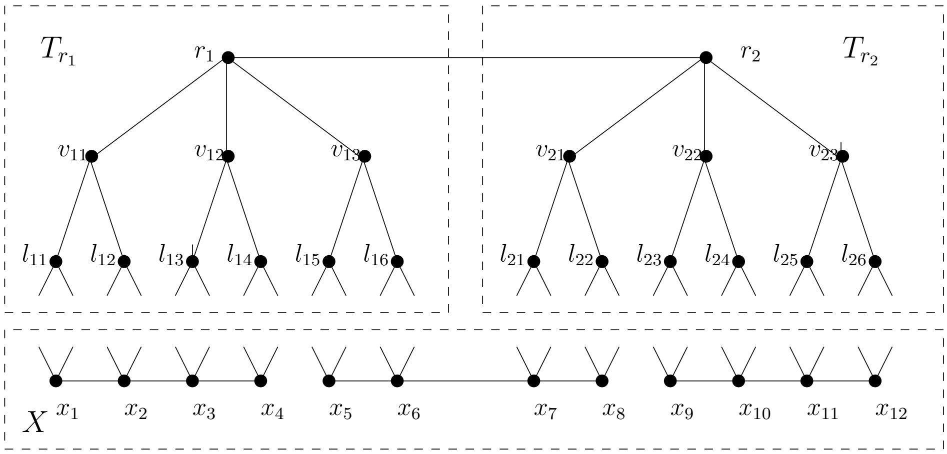


Figure 4.4: The trees T_{r_1} and T_{r_2} and the set X as described in Lemma 4.3 (iii).

Since $\mathcal{D} = (4^8, 3^{24})$ and $deg(r_1) = deg(r_2) = 4$, six of the vertices in X have degree 4 and the remaining 6 have degree 3. Since each x_i can only have 2 neighbours in $N_3(r)$, the other neighbours must be in X. The only way to create the required degrees on the vertices x_i is to have three paths of length 3 in X, that is, 4 vertices and three edges. But this will create cycles of length 5. Therefore, no two vertices of degree 4 can be neighbours.

(iv) Assume that there exists a vertex $r \in G$ such that $deg(r) = 3$ and $N(r) = \{v_1, v_2, v_3\}$ such that $deg(v_1) = deg(v_2) = deg(v_3) = 4$. Then T the tree with root r and height 3 is a subgraph of G. By item *(iii)* no two vertices of degree 4 are neighbours. Therefore, the neighbours of v_i have degree 3 and $|V(T)| = 1 + 3 + 9 + 18 = 31$ and $X = V(G - T) = \{x\}$. Then x has at most one neighbour in each set $N_3(r) \cap N_2(v_i)$, for $i = 1, 2$ and 3, otherwise a cycle of length 6 is formed. Thus, $deg(x) = 3$. Let l_i be a leaf of T such that $l_i \in N_3(r) \cap N_2(v_i)$. Then l_i has at most one neighbour in each set $N_3(r) \cap N_2(v_j)$ such that $i \neq j$ and one neighbour x. Since $deg(x) = 3$ there are at most 3 leaf vertices in T having degree 4. Then G contains at most 6 vertices of degree 4, namely, three leaf vertices that are neighbour of x and v_1, v_2, v_3. However, we know from *(ii)* that G contains eight vertices of degree 4. Therefore, a vertex of degree 3 has at most two neighbours having degree 4.

(v) By item *(ii)*, G contains eight vertices of degree 4. Assume that these vertices are distinct then G contains eight S_5 structures and $|V(G)| = 40$. Therefore, there are 8 vertices that are "shared" by the 5-stars. Items *(iii)* and *(iv)* assert that no two vertices of degree 4 are

neighbours and a vertex of degree 3 has at most two neighbours having degree 4. Then, by pigeonhole principle, every vertex of degree three has at least one neighbour of degree four and eight vertices of degree 3 have exactly two neighbours with degree 4.

(vi) By item (iii), no two vertices of degree 4 are neighbours. Item (v) establishes that there are eight vertices of degree 3 that have two neighbours having degree 4. Consequently, the eight vertices of degree 3 take $8 \times 2 = 16$ of the edges adjacent to vertices of degree 4. However, vertices of degree 4 are altogether adjacent to $8 \times 4 = 32$ vertices of degree 3. So there are 16 edges that have to be still allocated and they can only be connected to vertices of degree 3 which have only one degree 4 vertex in their neighbourhood.

(vii) By item (ii) there are twenty-four vertices of degree 3. Items (v) and (vi) assert that eight vertices of degree 3 have exactly two neighbours with degree 4 and sixteen vertices of degree 3 that have one neighbour having degree 4. Therefore, there are no vertices of degree 3 adjacent only to vertices of degree 3. ∎

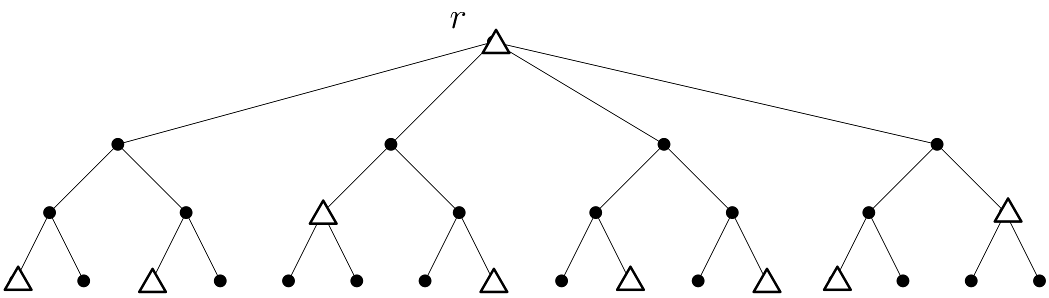


Figure 4.5: Illustration of the proof of Theorem 4.15.

◇ **Theorem 4.15** *Let* $G \in EX(32; 6)$. *Then* $ex(32; 6) = 51$.

Proof. We use the established upper and lower bounds given in Table 4.6, in particular, $51 \leq ex(32; 6) \leq 54$. From Theorem 4.15 we know that $ex(31; 6) = 49$. Applying Inequality 4.5,

$$ex(32; 6) - ex(31; 6) = ex(32; 6) - 49 \leq \delta \leq \lfloor (ex(32; 6) \times 2)/32 \rfloor.$$

The only possible values of $ex(32; 6)$ that satisfy this inequality are 51 and 52.

Assume $ex(32; 6) = 52$. Then applying Lemma 4.3 (iii) and (vii), no two vertices of degree 4 are neighbours and there are no vertices of degree 3 adjacent only to vertices of degree 3. That is, every vertex of degree 3 has at least one neighbour of degree 4. Figure 4.5 shows the tree with height 3, root r, such that $deg(r) = 4$, all of the vertices in the neighbourhood of r have degree 3 and every vertex of degree 3 has at least one neighbour of degree 4. The vertices of degree 4 are indicated by triangles in Figure 4.5. The tree has at least nine vertices of degree 4. However, by (*ii*) we know that G contains exactly eight vertices of degree 4.

Therefore, $ex(32; 6) = 51$. ■

A compilation of the current best known upper and lower bounds on $ex(n; 6)$, for $n \leq 200$ is given in Appendix B Table B.3. These bounds are not very tight, for example the differences between the upper and lower bound, for $n = 100$ and 200 are respectively, 36 and 114. These results demonstrate how dependent the GAP algorithm is on good seed graphs.

4.6.3 $ex(n; 7)$

The current known upper and lower bounds on $ex(n; 7)$ are contained in Table 4.8. Theorem 4.16 contains a construction by Abajo, Balbuena and Diánez [1] that uses a $(q + 1, 8)$-cage, where q is a prime power, to produce graphs that obtain good lower bounds on $ex(n; 7)$.

Theorem 4.16 *[1] Let $f_7(n) = ex(n; 7)$ and $q \geq 2$ be a prime power and $n_0(q+1, 8)$ the order of a (q+1,8)-cage. Then,*

$$f_7(n_0(q + 1, 8) + q^2 + q + 3) \geq f_7(n_0(q + 1, 8)) + 2(q^2 + q + 1) + 1.$$

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**9**
10	**10**	**12**	**13**	**14**	**16**	**18**	**19**	**20**	**22**	**24**
20	**25**	**27**	**29**	**30**	**32**	**34**	**36**	**38**	**40**	**42**
30	**45**	**46**	**47**	**49**	**51**	**53**	**55**	56-59	58-61	60-63
40	62-65	64-67	65-69	67-71	69-73	71-76	73-78	75-80	77-82	79-84
50	81-87	84-89	86-91	88-93	90-96	93-98	96-100	98-103	100-105	102-107
60	105-110	108-112	110-115	112-117	114-119	117-122	120-124	122-127	125-129	128-132
70	130-134	133-137	136-139	138-142	141-144	144-147	147-149	150-152	153-154	156-157
80	**160**									

Table 4.8: Known upper and lower bounds on $ex(n; 7)$, for $n \leq 19$ by [2] and, for $20 \leq n \leq 80$ by [3]. Exact values, when known, are listed in bold font.

The new lower bounds shown in Table 4.9 are generated from our GAP algorithm using the (4,8) and (5,8)-cages as seed graphs. The (4,8) and (5,8)-cages are Moore cages and therefore extremal, consequently, $ex(80; 7) = 160$ and $ex(170; 7) = 425$.

A construction due to Gács and Héger, [65] gives the following bounds, for prime powers q:

$n(q, 8) \leq 2(q^3 - 2q)$, for odd q
$n(q, 8) \leq 2(q^3 - 3q - 2)$, for even q
$n(q, 12) \leq 2(q^5 - q^3)$

When $q = 4$ this construction gives us the lower bound $ex(100; 7) \geq 200$. This construction has four more edges than the graph produced by our GAP algorithm using the Moore cages as input. So we used this value to improve lower bounds $ex(n; 7)$, for n close to 100.

n	0	1	2	3	4	5	6	7	8	9
80		161	162	164	166	168	170	172	174	176
90	178	180	181	183	185	187	189	191	193	196
100	200	201	202	204	206	209	212	215	218	221
110	224	227	230	233	236	239	242	245	248	252
120	256	258	261	264	267	270	273	276	278	281
130	284	287	290	293	296	299	302	306	309	312
140	316	319	322	325	328	332	335	338	342	345
150	348	352	356	360	363	367	371	374	378	382
160	385	389	393	396	400	404	408	412	416	420
170	**425**	426	427	429	431	433	435	436	438	440
180	442	444	446	448	450	452	453	455	457	459
190	461	463	465	467	469	471	473	475	477	479
200	481									

Table 4.9: ♢ New lower bounds on $ex(n; 7)$ produced by application of our GAP algorithm.

A compilation of the current best known upper and lower bounds on $ex(n; 7)$, for $n \leq 200$ is given in Appendix B Table B.4. Note that the difference between the current best known upper and lower bounds on $ex(n; 7)$ is comparable to those of $ex(n; 5)$, for example, $ex_u(100; 5) - ex_l(100; 5) = 14$ compared to $ex_u(100; 7) - ex_l(100; 7) = 13$ and $ex_u(200; 5) - ex_l(200; 5) = 46$ compared to $ex_u(200; 7) - ex_l(200; 7) = 33$. Furthermore, when $ex(n; 7)$ is known then $ex_u(n + 1; 7) - ex_l(n + 1; 7) \leq 3$ and $ex_u(n - 1; 5) - ex_l(n - 1; 5) = 1$.

4.7 $ex(n; t)$, for $t \in \{8, 9, 10, 11\}$

To date most research in the area of extremal graphs has focused on finding $ex(n; t)$ and graphs in $EX(n; t)$, for $t \leq 7$. Little is known about $ex(n; t)$ and the graphs in $EX(n; t)$, for $t \geq 8$, with the exception of the Moore cages (see Section 3.1.2) and the work by Abajo and Diánez [2] who established the extremal numbers, for $t \geq 4$ and $n \leq \lfloor (16t - 15)/5 \rfloor$ (see Section 4.5). The current known extremal numbers, for $t = 8, 9, 10$ and 11 due to Abajo and Diánez [2] are given in Tables 4.10, 4.12, 4.14 and 4.16, respectively.

Considering the evidence that the reliability of a network is enhanced when the smallest cycle in a graph is larger [98] prompted us to consider extremal graphs having larger values of t. Taking into account the relationship between cages and extremal graphs, we decided to limit our current investigation of extremal graphs to $G \in EX(n;t)$, for $t \leq 11$. In doing so, we use the known cages and graphs that give the current best known upper bounds on $n(k,g)$ (see Table 3.1) as seed graphs for our GAP algorithm.

In this section we establish the previously unknown values of $ex(n;8)$, for $n = 23, 24, 25, 26$; $ex(n;9)$, for $n = 26, 27, 28, 29$; and $ex(127;11)$. To do this we use the following lemma.

$\diamondsuit$ **Lemma 4.4** *There exists a graph $G \in EX(n;t)$ having minimum degree $\delta = 2$ and maximum degree $\Delta \geq 3$ and size $|E(G)| = ex(n;t) \in \{ex(n-1;t)+1, ex(n-1;t)+2\}$, when*

(i) $1 + 4\Delta \leq n < 46$, *for* $t = 8$.
(ii) $2 + 4\Delta \leq n < 62$, *for* $t = 9$.
(iii) $1 + 5\Delta \leq n < 94$, *for* $t = 10$.
(iv) $2 + 5\Delta \leq n < 126$, *for* $t = 11$.

Proof. From Theorem 4.3 there exists a graph with $\delta \geq 2$ and girth $g = t+1$, for $\mathrm{t}n > t + 1 + \lfloor (t-2)/2 \rfloor$. Applying the Moore bound for irregular graphs given in Theorem 3.2, for $ex(n < 46; 8)$, $ex(n < 62; 9)$, $ex(n < 94; 10)$ and $ex(n < 126; 11)$ determines upper and lower bounds on the average degree $\delta < \lfloor \overline{d} \rfloor < 3$. Therefore, there exists a graph with minimum degree $\delta = 2$ and girth $g = t+1$, in $EX(n < 46; 8)$, $EX(n < 62; 9)$, $EX(n < 94; 10)$ and $EX(n < 126; 11)$. Application of Theorem 4.1 gives the lower bounds on n in terms of the maximum degree, Δ. Furthermore, we establish $ex(n-1;t)+1 \leq ex(n;t) \leq ex(n-1;t)+\delta(G)$, for the given values of n and t by application of the Inequalities 4.2 and 4.4. ■

4.7.1 $ex(n;8)$

In Table 4.10 we provide a summary of the current known values of $ex(n;8)$. We make use of these values to determine the exact values of the extremal numbers $ex(23;8)$, $ex(24;8)$, $ex(25;8)$ and $ex(26;8)$. We then run our GAP algorithm to generate new lower bounds on $ex(n;8)$, for $n \leq 200$, given in Table 4.11.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**9**
10	**10**	**11**	**12**	**14**	**15**	**16**	**18**	**19**	**21**	**22**
20	**23**	**25**	**27**							

Table 4.10: Known values of $ex(n; 8)$, for $n \leq 22$, from [2].

$\diamondsuit$ **Theorem 4.17** *Let $G \in EX(23; 8)$. Then $ex(23; 8) = 28$.*

Proof. We know that $ex(22; 8) = 27$ (see Table 4.10). By Lemma 4.4 we have $ex(23; 8) \in \{28, 29\}$ and there exists a graph $G \in EX(23; 8)$ having $\delta = 2$, $\Delta \geq 3$ and $1 + 4\Delta \leq n = 23$. Therefore, $\Delta \leq 5$.

Assume $ex(23, 8) = 29$. From Table 4.10 we know $ex(21, 8) = 25$. Consequently, $ex(23; 8) - ex(21, 8) = 29 - 25 = 4$, thus, removing two vertices from G must remove at least 4 edges of G. Therefore, $P_2 = v_1v_2$ such that $deg(v_1) = deg(v_2) = 2$ is a forbidden subgraph of G and $T_{\Delta,2,8}(P_2^2)$ is a subgraph of G. Counting the vertices in $T_{\Delta,2,8}(P_2^2)$ (see Figure 4.6) we obtain

$$|V(G)| = 23 \geq |V(T_{\Delta,2,8}(P_2^2))| = 1 + \Delta + \Delta + 2\Delta + 2\Delta = 1 + 6\Delta.$$

Therefore, $\Delta = 3$.

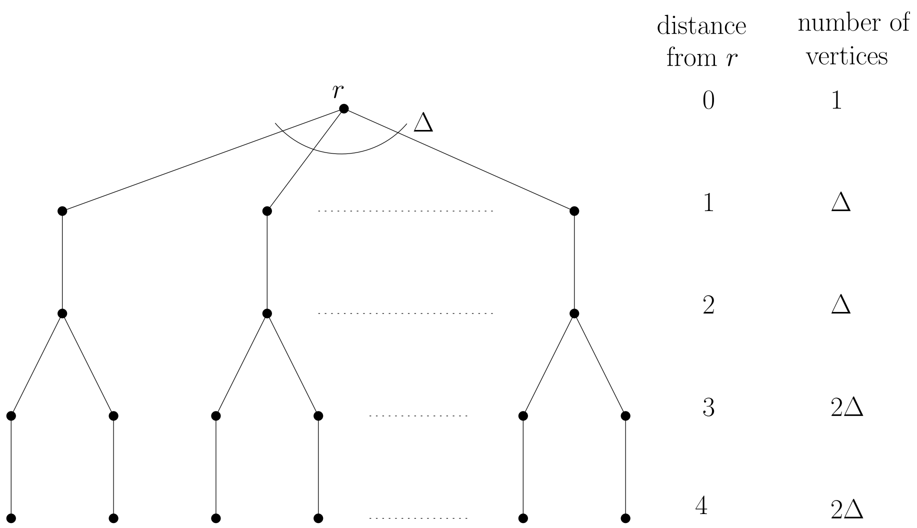


Figure 4.6: $|V(T_{\Delta,2,8}(P_2^2))| = 1 + 6\Delta$.

Let x be the number of vertices in G having degree 2, and y be the number of vertices of degree 3. Then $x + y = 23$ and $2x + 3y = 29 \times 2 = 58$. Solving these equations, we determine that there are twelve vertices of degree 3 and eleven vertices of degree 2 in G. Therefore, the degree sequence is $\mathcal{D}= (3^{12}, 2^{11})$.

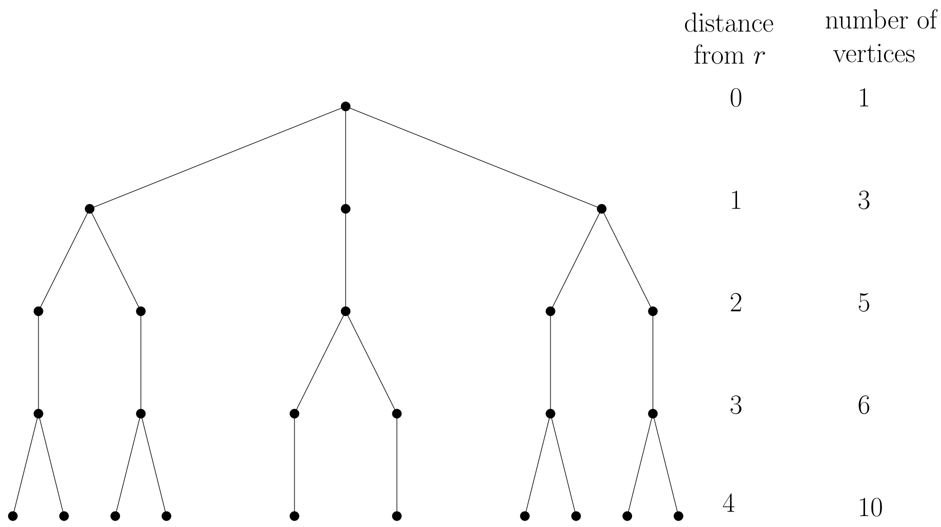


Figure 4.7: $T_{3,2,8}(P_2^2)$ having P_3^3 as a subgraph.

A vertex of degree 3 in G can have at most one neighbour having degree 3, otherwise $V|T_{3,2,8}(P_2^2)| \geq 25$ (see Figure 4.7). Therefore, the twelve vertices of degree 3 require twenty-four neighbours of degree 2. However, the eleven vertices of degree 2 have at most 22 neighbours of degree 3. Therefore, by pigeonhole principle, there is at least one vertex of degree 3 with more than one neighbour of degree 3. This is a contradiction. Therefore, $ex(23; 8) = 28$. ■

In Theorem 4.18, we use the results of Theorem 4.17 to determine the value of $ex(24; 8)$.

♢ **Theorem 4.18** *Let $G \in EX(24; 8)$. Then $ex(24; 8) = 29$.*

Proof. We know that $ex(23; 8) = 28$ from Theorem 4.17. By Lemma 4.4 we have $ex(24; 8) \in \{29, 30\}$ and there exists a graph $G \in EX(24; 8)$ having $\delta = 2$, $\Delta \geq 3$ and $1 + 4\Delta \leq n = 24$. Therefore, $\Delta \leq 5$.

Assume $ex(24; 8) = 30$. From Table 4.10, ex(21;8)=25. Consequently, $ex(24; 8) - ex(21; 8) = 30 - 25 = 5$, thus, removing three vertices from G must cause the removal of at least five edges from G and P_3^2 is a forbidden subgraph.

From Table 4.10, ex(20;8)=23. Therefore, $ex(24; 8) - ex(20; 8) = 30 - 23 = 7$. Consequently, removing four vertices from G must result in the removal of at least seven edges from G. Therefore, G contains at most one path, P_2^2, of length 2 such that all vertices in the path have degree 2 and $2P_2^2$ is a forbidden subgraph of G.

Let $\mathcal{F}= \{2P_2^2, P_3^2\}$ represent these forbidden subgraphs. Then the minimal tree $T_{\Delta,2,8}(\mathcal{F})$ has order $6\Delta \leq |V(G)| = 24$. Therefore, $\Delta \leq 4$.

Assume $\Delta = 4$. Then $|V(T_{\Delta,2,8}(\mathcal{F}))| = 24 = |V(G)|$ as shown in Figure 4.8. Therefore, $T_{4,2,8}(\mathcal{F})$ is a spanning tree of G.

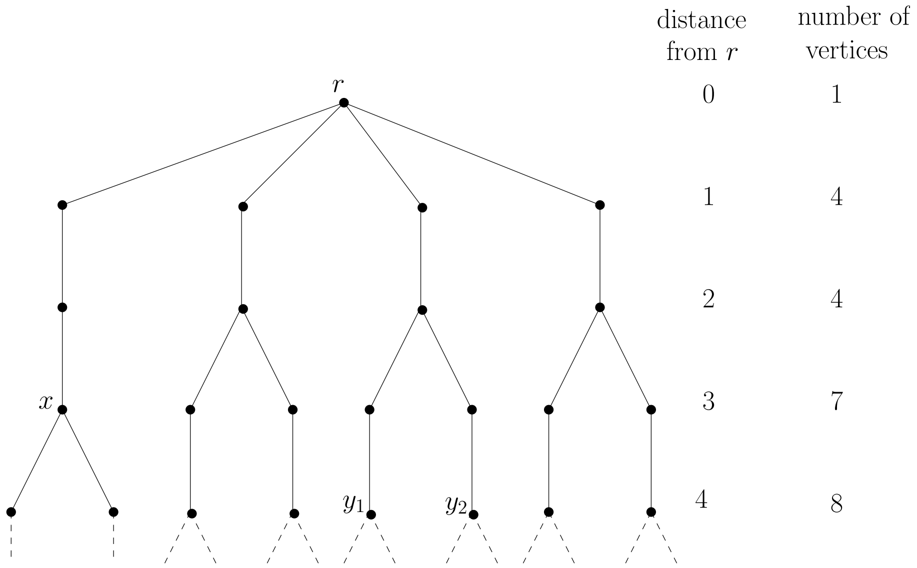
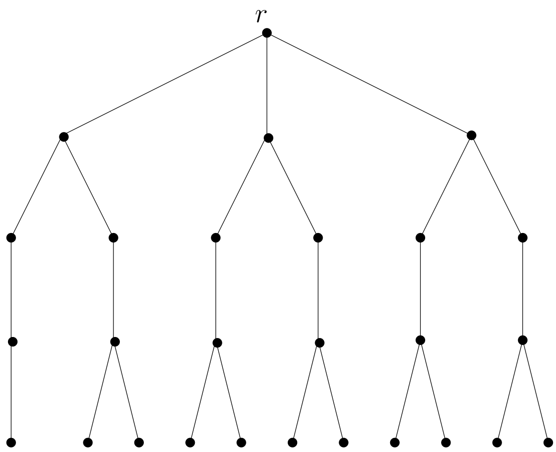


Figure 4.8: $|V(T_{4,2,8}(\ \mathcal{F}\ ))| = 24$.

By our assumption, $|E(G)| = 30$. Therefore, $|E(G) - E(T_{4,2,8}(\mathcal{F}))| = 30 - 23 = 7$. Consequently, there are 7 edges between the leaves of $T_{4,2,8}(\mathcal{F})$. Furthermore, in order to comply with the forbidden subgraphs constraints, the eight leaves of $T_{4,2,8}(\mathcal{F})$ must have degrees (2,2,3,3,3,3,3,3) in G as indicated by the edges in the tree and dashed lines shown in Figure 4.8. Assume that these seven edges can be added between the eight leaves of $T_{4,2,8}(\mathcal{F})$ without forming a short cycle. Then there is a cycle, C_9, of length 9 that includes the eight leaves of $T_{4,2,8}(\mathcal{F})$ and the vertex x (see Figure 4.8). Consider any two vertices that are in the same branch of the tree $T_{4,2,8}(\mathcal{F})$, for example, y_1 and y_2 as shown in Figure 4.8. The distance between y_1 and y_2 via $T_{4,2,8}(\mathcal{F})$ is 4 and the distance between y_1 and y_2 via C_9 is at most 4. Therefore, a forbidden cycle of length at most 8 exists through the vertices y_1 and y_2. Consequently, $\Delta = 3$.

Let x be the number of degree 3 vertices in G and y the number of vertices of degree 2. Then $x + y = 24$ and $3x + 2y = 30 \times 2$. Solving these equations determines that there are twelve vertices of degree 3 and twelve vertices of degree 2. Therefore, the degree sequence of G is $\mathcal{D}= (3^{12}, 2^{12})$.

The tree $T_{3,2,8}(\mathcal{F})$, with the additional constraint that r has three neighbours of degree 4, illustrated in Figure 4.9, has 27 vertices. Therefore, any vertex $v \in G$ with $deg(v) = 3$ has at most two neighbours of degree 3.



distance from r	number of vertices
0	1
1	3
2	6
3	6
4	11

Figure 4.9: $|V(T_{3,2,8}(\ \mathcal{F}\ ))| = 27$.

We now resolve the structure of the graph and the lengths of the cycles on which the vertices of degree 3 lie. Recall, $\mathcal{D}=(3^{12},2^{12})$. Let x_i be the vertices of degree 2 and y_i the vertices of degree 3, where $i = 1, 2, \ldots, 12$. Since $2P_2^2 \in \mathcal{F}$ any path P_4 must contain at least two vertices of degree 3. Assume that G contains one path $P_2^2 = \{x_12, x_1\}$. Then all other vertices of degree 2 in G have two neighbours of degree 3 and G is composed of either a cycle $C_{23} = x_1, y_1, x_2, y_2, \ldots, x_{11}, y_{11}, x_{12}, x_1$ and the vertex y_{12} or the cycles $C_{11} = x_{12}, x_1, y_1, x_2, y_2, \ldots, x_5, y_5, x_{12}$ and $C_{12} = x_6, y_6, x_7, y_7, \ldots, x_{11}, y_{11}$ and the vertex y_{12}. In either cases the vertex y_{12} must have three neighbours of degree 3, which is a contradiction.

Therefore, G contains either one cycle $C_{24} = x_1, y_1, x_2, y_2, \ldots, x_{12}, y_{12}, x_1$ or two cycles $C_{12} = x_1, y_1, x_2, y_2, \ldots, x_6, y_6, x_1$ and $C_{12} = x_7, y_7, x_8, y_8, \ldots, x_{12}, y_{12}, x_7$. Both scenarios have 24 vertices and 24 edges. We have to add another six edges between the vertices y_i, for $i = 1, 2 \ldots, 12$. We can not add six edges into C_{24} without creating a forbidden cycle. No edges can be placed into a C_{12} without creating forbidden cycles and it is not possible to place six edges between the two cycles of length 12 without constructing cycles of length less than nine. This can be demonstrated as follows. Without loss of generality add the edge y_1y_7 between the two C_{12} cycles, then the only vertex that can be adjacent to y_2 without creating forbidden cycles is y_{10}. Then y_6 must be adjacent to one of the vertices in the set $\{y_8, y_9, y_{11}, y_{12}\}$. Creating forbidden cycles.

Therefore, $ex(24; 8) = 29$. $\blacksquare$

The construction in the proof of the following theorem was obtained using Exoo's [53] randomised hill-climbing back-tracking algorithm. We used this algorithm after a number of failed attempts to prove that $ex(25; 8) = 30$.

$\diamondsuit$ **Theorem 4.19** *Let* $G \in EX(25; 8)$. *Then* $ex(25; 8) = 31$.

Proof. We know that $ex(24; 8) = 29$ by Theorem 4.18. From Lemma 4.4 $ex(25; 8) \in \{30, 31\}$ and there exists a graph G with minimum degree $\delta = 2$, maximum degree $\Delta \geq 3$ and $1+4\Delta \leq n$.

We have constructed a graph of order G, size 31, girth 9, degree sequence $\mathcal{D} = (3^{12}, 2^{13})$ and edge set, $E(G) = \{\{1, 6\}, \{1, 22\}, \{2, 14\}, \{2, 20\}, \{3, 5\}, \{3, 14\}, \{3, 24\}, \{4, 6\}, \{4, 11\}, \{4, 19\}, \{5, 19\}, \{6, 23\}, \{7, 13\}, \{7, 15\}, \{7, 16\}, \{8, 15\}, \{8, 22\}, \{8, 24\}, \{9, 11\}, \{9, 12\}, \{9, 21\}, \{10, 12\}, \{10, 14\}, \{10, 18\}, \{13, 20\}, \{15, 21\}, \{16, 17\}, \{16, 25\}, \{17, 19\}, \{18, 25\}, \{20, 23\}\}$.

Therefore, $ex(25; 8) = 31$. ■

The construction in the proof of the following theorem was obtained using our GAP algorithm with the graph given in the proof of Theorem 4.19 as a seed graph.

$\diamondsuit$ **Theorem 4.20** *Let* $G \in EX(26; 8)$. *Then* $ex(26; 8) = 33$.

Proof. We know that $ex(25; 8) = 31$ from the construction in Theorem 4.19. From Lemma 4.4 we know $ex(26; 8) \in \{32, 33\}$, and there exists a graph G with minimum degree $\delta = 2$, maximum degree $\Delta \geq 3$ and $1 + 4\Delta \leq n$.

Adding one vertex $\{26\}$ and two edges $\{\{1, 26\}, \{18, 26\}\}$ to the graph constructed in the proof of Theorem 4.19 results in a graph of order 26, size 33 and girth 9.

Therefore, $ex(26; 8) = 33$. ■

We ran our GAP algorithm using the construction from Theorem 4.20 and the (3,9)-cage as input graphs. The new lower bounds on $ex(n; 8)$ that were obtained form running our GAP algorithm are given in Table 4.11

4.7.2 $ex(n; 9)$

In Table 4.12 we provide a summary of the current known values of $ex(n; 9)$. We make use of these values to determine the exact values of the extremal numbers $ex(26; 9)$, $ex(27; 9)$, $ex(28; 9)$

n	0	1	2	3	4	5	6	7	8	9
20				**28**	**29**	**31**	**33**	34	35	37
30	39	40	42	43	45	47	48	50	52	54
40	55	57	58	60	62	64	65	67	69	71
50	73	75	77	78	80	81	83	85	87	88
60	90	91	93	95	97	99	100	102	103	105
70	107	108	110	112	114	115	117	118	120	122
80	124	125	127	129	131	133	134	136	138	140
90	141	143	145	147	149	151	153	155	157	159
100	161	163	164	166	168	170	172	173	175	177
110	179	181	183	185	187	188	190	192	194	196
120	198	200	202	204	206	208	210	212	213	215
130	217	219	221	223	225	227	229	231	233	235
140	237	239	241	243	245	247	249	251	253	255
150	257	259	261	263	265	267	269	272	274	275
160	277	279	281	283	285	287	289	291	293	295
170	297	299	301	303	305	307	309	311	313	315
180	317	319	321	323	325	327	329	331	333	335
190	337	339	341	343	345	347	349	351	353	355
200	357									

Table 4.11: ♢ New lower bounds on $ex(n;8)$, for $n \leq 200$, produced by application of our GAP algorithm.

and $ex(29;9)$. We then run our GAP algorithm to generate new lower bounds on $ex(n;9)$, for $n \leq 200$, as shown in Table 4.13.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**10**	**11**	**12**	**13**	**15**	**16**	**17**	**18**	**20**	**21**
20	**23**	**24**	**25**	**27**	**28**	**30**				

Table 4.12: Known values of $ex(n;9)$, for $n \leq 25$, from [2].

In the following theorem we determine the extremal number, for the smallest value of n that is not yet known, namely, $ex(26;9)$.

♢ **Theorem 4.21** *Let $G \in EX(26;9)$. Then $ex(26;9) = 31$.*

Proof. We know that $ex(25;9) = 30$ (see Table 4.12). By Lemma 4.4 we have $ex(26;9) \in \{31, 32\}$ and there exists a graph $G \in EX(26;9)$ having $\delta = 2$, $\Delta \geq 3$ and $2 + 4\Delta \leq n = 26$. Therefore, $\Delta \leq 6$.

Assume $ex(26;9) = 32$. Then, since $ex(24;9) = 28$ (see Table 4.12), removing two vertices must remove at least four edges. Therefore, the path P_2^2 is a forbidden subgraph of G and $|V(T_{\Delta,2,9}(P_2^2))| = 6\Delta + 5 \leq |V(G)| = 26$ as illustrated in Figure 4.10. Therefore, $\Delta = 3$.

Let x be the number of vertices of degree 3 in G and y the number of vertices of degree 2. Then $x + y = 26$ and $3x + 2y = 32 \times 2 = 64$. Solving these equations we resolve the degree sequence of G to be $\mathcal{D} = (3^{12}, 2^{14})$.

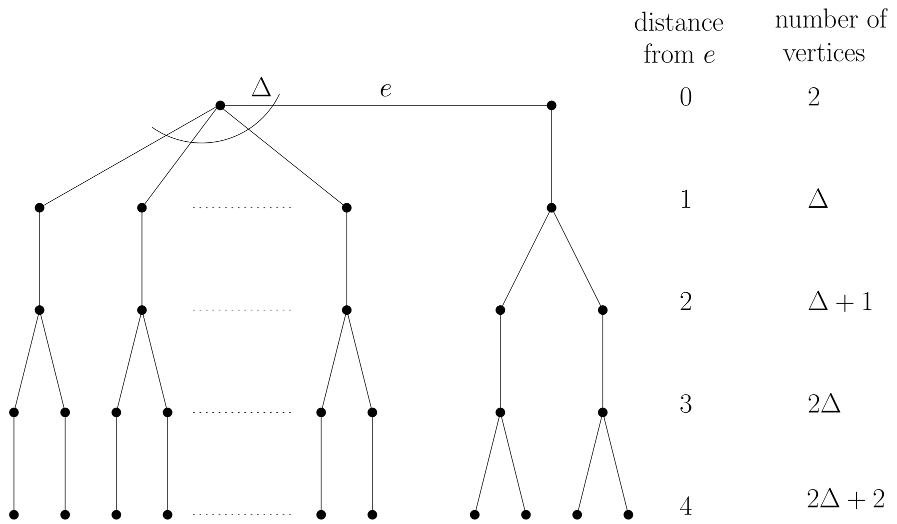
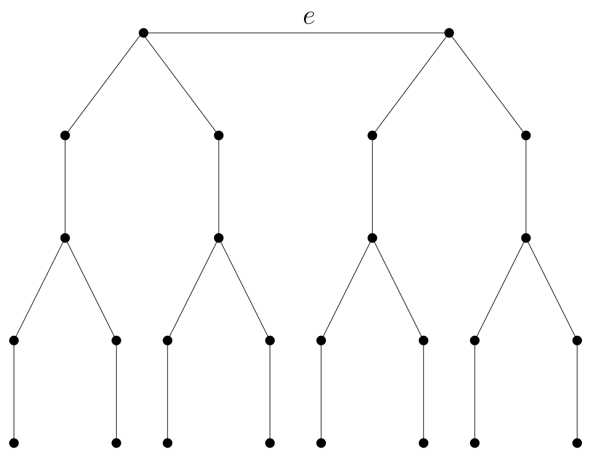


Figure 4.10: $|V(T_{\Delta,2,9}(P_2^2))| = 6\Delta + 5$.

Next we prove that every neighbour of a vertex v such that $deg(v) = 3$ must have degree 2. Let $e = r_1r_2$ be an edge such that $deg(r_1) = deg(r_2) = 3$. Then $|V(T_{3,2,9}(P_2^2))| = 26$, as shown in Figure 4.11. Note that if a neighbour of r_1 has degree 3 then the minimal tree with these extra constraints contains at least 27 vertices. Therefore, a vertex of degree 3 has at most one neighbour of degree 3. Since P_2^2 is a forbidden subgraph of G, the leaves of the tree $T_{3,2,9}(P_2^2)$ must have degree 3 in G and there are 8 edges between the leaves of $T_{3,2,9}(P_2^2)$, thus creating a forbidden cycle of length at most 8. Therefore, every neighbour of a vertex of degree 3 must have degree 2. Since there are twelve vertices of degree 3, we require $12 \times 3 = 36$ edges xy such that $deg(x) = 3$ and $deg(y) = 2$. However, there are only fourteen vertices of degree 2 providing $14 \times 2 = 28$ available edges.

Therefore, $ex(26;9) = 31$. ■



distance from e	number of vertices
0	2
1	4
2	4
3	8
4	8

Figure 4.11: $|V(T_{3,2,9}(P_2^2))| = 26$.

$\diamondsuit$ **Theorem 4.22** *Let $G \in EX(27; 9)$. Then $ex(27; 9) = 32$.*

Proof. From Theorem 4.21 we know $ex(26; 9) = 31$. By Lemma 4.4 we have $ex(27; 9) \in \{32, 33\}$ and there exists a graph $G \in EX(27; 9)$ having $\delta = 2$, $\Delta \geq 3$ and $2 + 4\Delta \leq n = 27$. Therefore, $\Delta \leq 6$.

Assume $ex(27; 9) = 33$. From Table 4.12 we know $ex(24; 9) = 28$. Consequently, removing 3 vertices from G must remove at least 5 edges and P_3^2, is a forbidden subgraph. Then $|V(T_{\Delta,2,9}(P_3^2))| = 3 + 5\Delta < |V(G)| = 27$ (see Figure 4.12). Therefore, $\Delta \leq 4$.

Assume $\Delta = 4$. Then $|V(X)| = |V(G) - V(T_{4,2,9}(P_3^2)))| = 27 - 23 = 4$. Let $X = \{x_1, x_2, x_3, x_4\}$.

Assume that there are two vertices r_1 and r_2 of degree 4 that are neighbours. Then the minimal tree $T_{4,2,9}(P_3^2)$ with this additional constraint has order $|V(T_{4,2,9}(P_3^2))| = 2+6+6+6+12 = 32$. Therefore, no two vertices of degree 4 are neighbours.

Assume that a vertex r_1 of degree 4 is adjacent to a vertex r_2 of degree 3. Then the minimal tree $T_{4,2,9}(P_3^2)$ with this additional constraint has order $|V(T_{4,2,9}(P_3^2))| = 2+5+5+5+10 = 27$. Therefore, $T_{4,2,9}(P_3^2)$ is a spanning tree of G. There are 26 edges in $T_{4,2,9}(P_3^2)$. There must be 33-26=7 edges between the leaves of the tree. Assume that we can place the required seven edges between the leaves of the tree. Then consider the ten leaves and their five parent vertices as a subgraph of G. These fifteen vertices must have 17 edges between them. However, from Table 4.12 we known that $ex(15; 9) = 16$. Therefore, all neighbours of a vertex of degree 4 have degree 2. We constructed all possible graphs, considering these constraints and found that

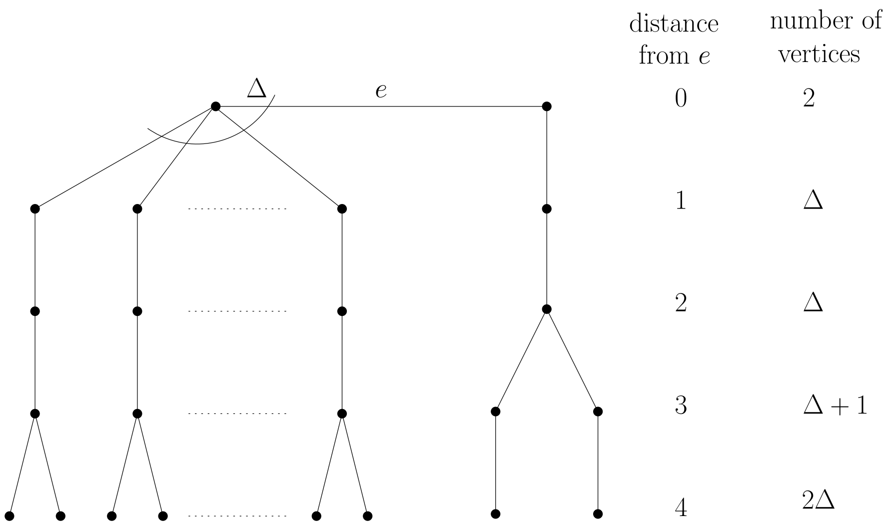


Figure 4.12: $|V(T_{\Delta,2,9}(P_3^2))| = 3 + 5\Delta$.

there can be at most 32 edges in G. Two of the graphs with 32 edges had degree sequences $\mathcal{D} = (4^1, 3^8, 2^{18})$ and $\mathcal{D} = (4^3, 3^4, 2^{20})$.

Therefore, $\Delta = 3$. Let x denote the number of vertices of degree 3 in G and y the number of vertices of degree 2. Then $x + y = 27$ and $3x + 2y = 33 \times 2 = 66$. Solving these equations resolves the degree sequence of G to be $\mathcal{D} = (3^{12}, 2^{15})$.

The fact that P_3^2 is a forbidden subgraph means that every vertex of degree 2 has at least one neighbour or degree 3. Let x_i, for $i = 1, 2, \ldots, 15$ denote the vertices of degree 2 and y_j, for $j = 1, 2, \ldots, 12$ denote the vertices of degree 3. Let $x_i y_j$, for $i = j = 1, 2, \ldots, 12$ denote these edges. Then there is a cycle of length 15 through the twelve vertices of degree 3 and three vertices of degree 2, namely, x_{13}, x_{14} and x_{15}. The other twelve vertices of degree 2 are pendant vertices to the twelve vertices of degree 3 as shown in Figure 4.13. Although the placement of the vertices x_{13}, x_{14} and x_{15} on the cycle is somewhat arbitrary, each of the vertices x_{13}, x_{14} and x_{15} have two neighbours of degree 3 on the cycle. The graph in Figure 4.13 has 27 vertices and 27 edges. It is impossible to add another six edges between the pendant vertices x_i, for $i = 1, 2, \ldots, 12$.

Therefore, $ex(27; 9) = 32$. Four graphs that reach this value have degree sequences $\mathcal{D} = (3^{11}, 2^{15}, 1^1)$, $\mathcal{D} = (3^{10}, 2^{17})$, $\mathcal{D} = (4^1, 3^8, 2^{18})$ and $\mathcal{D} = (4^3, 3^4, 2^{20})$. ∎

In the following two theorems we establish the values of $ex(28; 9)$ and $ex(29; 9)$. The constructions described in the proofs of these theorems were obtained by our GAP algorithm using the subdivided Petersen graph $s_1 P \in EX(25; 9)$ as input.

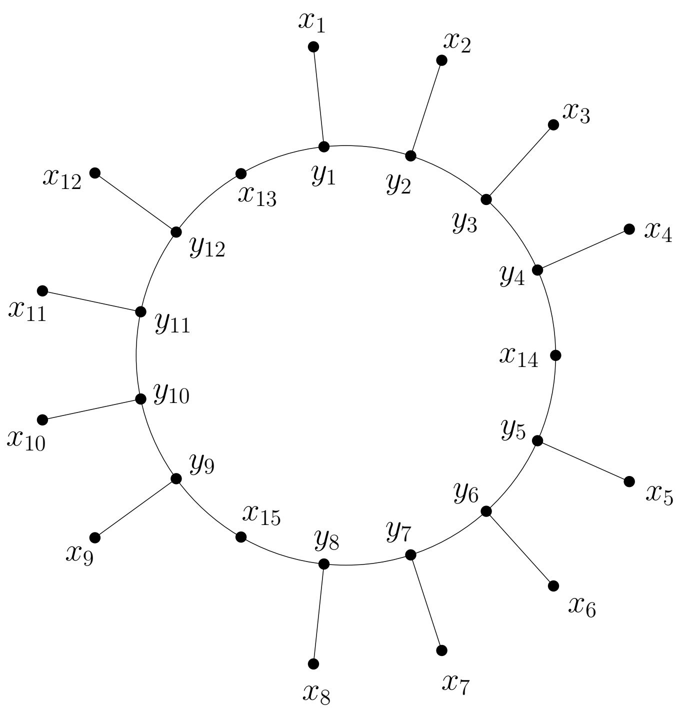


Figure 4.13: The structure of a graph with $\mathcal{D} = (3^{12}, 2^{15})$, when P_3^2 is a forbidden subgraph.

$\diamondsuit$ **Theorem 4.23** *Let* $G \in EX(28; 9)$*. Then* $ex(28; 9) = 34$*.*

Proof. From Theorem 4.22 we know $ex(27; 9) = 32$. By Lemma 4.4 we have $ex(28; 9) \in \{33, 34\}$. Recall from Theorem 4.7 the subdivided Petersen graph is extremal, that is, $s_1P \in EX(25; 9)$ and $ex(25; 9) = 30$. The diameter of s_1P is 6.

Given $s_1P \in EX(25; 9)$ and three vertices $\{x_1, x_2, x_3\}$, such that $x_i \notin s_1P$, for $i = 1, 2, 3$, we can create a new graph G with $V(G) = V(s_1P) \cup \{x_1, x_2, x_3\}$ and $E(G) = E(s_1P) \cup \{ux_1, x_1x_2, x_2x_3, x_3v\}$, where u and v are vertices in s_1P, such that $d(u, v) = 6$. G has order 28, size 34 and girth 10.

Therefore, $ex(28; 9) = 34$. ■

$\diamondsuit$ **Theorem 4.24** *Let* $G \in EX(29; 9)$*. Then* $ex(29; 9) = 36$*.*

Proof. From Theorem 4.23 we know $ex(28; 9) = 34$. By Lemma 4.4 we have $ex(28; 9) \in \{35, 36\}$. The construction of the graph $G \in EX(28, 9)$ described in the proof of Theorem 4.23 has diameter 8. Therefore, given G and a vertex $x \notin V(G)$ we can create a new graph G' with

$V(G') = V(G) \cup \{x\}$ and $E(G') = E(G) \cup \{ux, xv\}$, where u and v be vertices in G, such that $d(u, v) = 8$. The new graph G' has order 29, size 36 and girth 10.

Therefore, $ex(29; 9) = 36$. ■

We used the constructions in the proofs of Theorems 4.22 and 4.23 and the (3,10)-cage as seed graphs to our GAP algorithm. The (3,10)-cage has order 70 and gives the lower bound on $ex_l(70; 9) = 105$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
20							**31**	**32**	**34**	**36**
30	37	38	40	42	43	44	46	48	49	50
40	52	54	55	57	59	60	61	63	64	66
50	68	70	71	72	74	76	78	79	81	83
60	85	86	88	90	92	94	96	98	100	102
70	105	106	107	109	110	112	114	115	116	118
80	120	121	123	124	126	128	129	131	133	134
90	136	138	139	141	143	144	146	148	149	151
100	152	154	156	158	159	161	163	165	167	168
110	170	172	174	176	178	179	181	183	185	186
120	188	189	191	193	195	196	198	200	202	203
130	205	207	208	210	212	214	216	218	220	222
140	224	226	228	230	232	234	235	237	239	241
150	243	246	247	248	249	250	252	254	256	258
160	259	261	263	265	267	269	271	273	275	277
170	279	281	283	285	287	289	291	293	295	297
180	299	301	303	305	307	308	310	312	314	316
190	318	320	322	324	326	328	330	332	334	336
200	338									

Table 4.13: ♢ New lower bounds on $ex(n; 9)$, for $n \leq 200$, produced by application of our GAP algorithm.

4.7.3 $ex(n; 10)$

We use the (3,11)-cage as a seed graph to our GAP algorithm. The (3,11)-cage is has order 112 and size 168. Applying the lower bound on $ex(112; 10)$ due to the (3,11)-cage and the upper bound on $ex(112; 10)$ due to the Moore bound for irregular graphs, we know, $168 \leq ex(112; 10) \leq 173$. The new current best lower bounds on $ex(n; 10)$ produced by our GAP algorithm are displayed in Table 4.15.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**9**	**11**	**12**	**13**	**14**	**15**	**17**	**18**	**19**	**20**
20	**22**	**23**	**24**	**26**	**27**	**28**	**30**	**31**	**33**	**34**

Table 4.14: Known values of $ex(n;10)$, for $n \leq 29$, from [2].

Abajo, Balbuena and Diánez [1] constructed some infinite families to provide better lower bounds on $ex(n;10)$ as shown in Theorem 4.25. In particular, $ex(112;10) > 168$, $ex(116;10) > 174$, $ex(120;10) > 180$, and $ex(122;10) > 183$. We obtained the same lower bounds obtained using our GAP algorithm, for $n = 112, 116, 120$ and 122. These values are displayed in italic font in Table 4.15.

Theorem 4.25 *[1] Let $f_{10}(n) = ex(n;10)$ and $q \geq 3$ be a prime power. Then*

(i) $f_{10}(2q^5 + 2q^4 + 2q^3 + 2q^2 + q) \geq q^2(q+1)^2(q^2+1) + (q+1)f_{10}(q)$.
(ii) $f_{10}(2(q^5 + q^4 + q^3 + q^2)) \geq q^3(q^3 + 2q^2 + 2q + 2) + 2qf_{10}(q)$..
(iii) $f_{10}(2q^5 + 2q^4 + 2q^3 + q^2) \geq q^3(q+1)(q^2+q+1) + q(q+1)f_{10}(q)$.
(iv) $f_{10}(2(q^5 + q^4 + q^3)) \geq q^6 + 2q^5 + 2q^4 + 2q^2 f_{10}(q)$.
(v) $f_{10}(2(q^5 + q^4 + q^3) - h) \geq q^6 + 2q^5 + 2q^4 + 2q^2 f_{10}(q) - h(q+1)$*, for all* $h = 1, \ldots, q^5$.

4.7.4 $ex(n;11)$

Finally, for $t = 11$, we ran our GAP program with seed graphs being: the subdivided Heawood graph s_1H; the subdivided complete bipartite graph $s_2K_{5,5}$; and the $(3,12)$-cage, otherwise known as the Benson graph. The fact that s_1H is extremal and $ex(35,11) = 42$, was established in Section 4.5. The graph $s_2K_{5,5}$ gives the lower bound $ex(60;11) \geq 75$. The Benson graph, having 126 vertices and 189 edges, is the $(3,12)$-Moore cage and is therefore extremal and $ex(126;11) = 189$.

In the following theorem we use the fact that the Benson graph is extremal and apply the upper bound on the extremal number due to the Moore bound for irregular graphs, to establish the previously unknown extremal number $ex(127;11) = 190$.

$\diamondsuit$ **Theorem 4.26** *$ex(127;11) = 190$ and there exists a graph $G \in EX(127;11)$ with minimum degree 1.*

Proof. We know that $ex(126;11) = 189$ due to the (3,12)-cage being the Moore cage on 126 vertices. We can add one vertex and one edge to the (3,12)-cage, without reducing the girth, by either adding a pendant vertex or by subdividing an arbitrary edge. Therefore, by construction,

n	0	1	2	3	4	5	6	7	8	9
30	35	36	38	39	41	42	43	44	46	47
40	49	50	52	53	54	56	57	59	60	62
50	63	65	66	67	69	71	72	73	75	76
60	78	79	81	82	84	86	87	88	90	91
70	93	94	96	97	99	100	102	103	105	107
80	108	110	111	113	114	116	117	119	121	123
90	125	127	129	130	132	134	136	138	140	142
100	144	146	147	149	151	153	155	157	159	161
110	163	165	*168*	169	171	172	*174*	175	177	178
120	*180*	181	*183*	184	186	188	189	190	192	193
130	195	196	198	199	201	202	204	205	207	208
140	210	211	213	214	216	217	219	220	222	223
150	225	226	228	229	231	232	234	236	238	239
160	240	242	244	245	247	248	250	251	253	255
170	256	258	259	261	262	264	266	267	269	270
180	272	273	275	276	278	279	281	282	284	285
190	287	289	291	292	294	295	297	298	300	302
200	303									

Table 4.15: $\diamondsuit$ New lower bounds on $ex(n;10)$, for $n \leq 200$, produced by application of our GAP algorithm.

$ex(127;11) \geq ex(126;11)+1 = 190$ and there exists G such that $\mathcal{D}(G) = (3^{126}, 2^1)$ and G' such that $\mathcal{D}(G') = (4^1, 3^{125}, 1^1)$.

Assume $ex(127;11) = 191$ and $G \in EX(127;11)$. Then the average degree is $\overline{d} = (191 \times 2)/127$. Applying the Moore bound for irregular graphs with average degree $\overline{d} = (191 \times 2)/127$ and girth $g = 12$ determines that G has at least 128 vertices which is a contradiction. Therefore, $ex(127;11) = 190$. $\blacksquare$

Some of the current best known lower bounds on $ex(n;11)$ are the trivalent symmetric graphs found by Condor and Dobcsányi [40], namely, for $n = 162, 168, 182, 192, 204$ and 216. These graphs have diameter 8, for $n = 162, 168, 192, 216$ and diameter 9, for $n = 182$ and 204. Using the knowledge of the diameter of these graphs we manually calculate the lower bounds on $ex(n;11)$, for some other values of n close to these values.

Theorem 4.27 *[1] Let q be an odd prime power different from 5 and 7 and let $n_0(q+1,12)$ denote the order of a minimal $(q+1,12)$-cage. Then,*

$$f_{11}(n_0(q+1,12)+2q^3+2q+5) \geq f_{11}(n_0(q+1,12))+3q^3+3q+5.$$

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**9**	**10**	**12**	**13**	**14**	**15**	**16**	**18**	**19**	**20**
20	**21**	**22**	**24**	**25**	**27**	**28**	**29**	**30**	**32**	**33**
30	**34**	**36**	**37**							

Table 4.16: Known values of $ex(n;11)$, for $n \leq 32$, from [2].

n	0	1	2	3	4	5	6	7	8	9
30				38	40	**42**	43	44	45	46
40	48	49	50	52	53	54	56	57	58	60
50	61	62	64	66	67	68	70	71	72	74
60	75	80	81	82	84	85	87	88	89	91
70	92	94	95	97	98	100	101	103	104	106
80	107	109	110	112	113	115	116	118	119	121
90	122	124	125	127	128	130	132	133	135	136
100	138	139	141	143	145	147	149	151	153	155
110	157	159	161	163	165	166	168	170	172	174
120	176	178	180	182	184	186	**189**	**190**	191	192
130	193	195	196	198	199	200	202	203	205	206
140	207	209	210	212	213	214	216	217	219	220
150	221	223	224	226	227	229	230	232	234	236
160	238	240	243	244	245	247	248	249	252	253
170	254	256	257	258	259	260	261	262	264	266
180	268	270	273	274	276	277	278	279	280	281
190	283	285	288	289	290	292	293	294	295	296
200	298	300	302	303	306	307	309	310	311	312

Table 4.17: $\diamondsuit$ New lower bounds on $ex(n;11)$, for $n \leq 209$, produced by application of our GAP algorithm.

Compilations of the current best known upper and lower bounds on $ex(n;t)$, for $n \leq 200$ and $t = 8, 9, 10$ and 11 are given in Appendix B Tables B.5, B.6, B.7 and B.8.

In Table 4.18 we compare the differences between $ex_u(n;t)$ and $ex_l(n;t)$, for $n = 50, 100, 150$ and 200 and $4 \leq t \leq 11$.

t	4	5	6	7	8	9	10	11
$ex_u(50;t) - ex_l(50;t)$	0	35	7	6	3	3	4	3
$ex_u(100;t) - ex_l(100;t)$	54	14	36	13	18	13	7	6
$ex_u(150;t) - ex_l(150;t)$	117	19	48	13	38	28	19	10
$ex_u(200;t) - ex_l(200;t)$	226	46	114	43	63	46	39	25

Table 4.18: The difference between $ex_u(n;t)$ and $ex_l(n;t)$, for $n = 50, 100, 150$ and 200 and $4 \leq t \leq 11$.

Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

Alfred North Whitehead

5

Connectivity

In this chapter, we present our results on the connectivity, more precisely, vertex connectivity, of a graph. Recalling from Section 2.5, we say that G is r-*connected* if the deletion of at least r vertices of G is required to disconnect the graph. Furthermore, the vertex connectivity, denoted by κ, of a connected graph G is the smallest number of vertices whose removal disconnects G. More formally,

$$\kappa = min\{|X| : X \subseteq V(G) \text{ and } \omega(G - X) > 1\},$$

where $\omega(G - X)$ is the number of components of the graph obtained from G by removing the vertices of X.

Our first theorem, presented in Section 5.1, is an extension of a Theorem by Balbuena and Marcote [18], which was given in Section 3.3, namely, any r-regular graph G with $r \geq 2$ and girth g has vertex connectivity,

$$\kappa \geq 2 \text{ if } \begin{cases} D \leq 2\lfloor (g-1)/2 \rfloor + 2, & \text{for } r \leq 3 \\ D \leq 2\lfloor (g-1)/2 \rfloor + 1, & \text{for } r > 3 \end{cases}$$

We extend this result to include non regular graphs, by showing that a non regular graph G with minimum degree $\delta \geq 2$ and diameter $D \leq g - 1$ is 2-connected if the girth is even or $\Delta \leq 2\delta - 1$, when the girth is odd.

Moreover, in Section 5.2, we improve upon a result by Balbuena, Carmona, Fàbrega and Fiol [11], which was stated in Section 3.3, that is, a graph G with minimum degree $\delta \geq 2$, girth g, edge minimum degree ξ has vertex connectivity,

$$\kappa \geq \min\{\delta, 4\} \text{ if } D \leq g-2, g \text{ odd} .$$

We improve upon this result by proving that any graph G having even girth and $\Delta \leq 2\delta - 5$ is 5-connected.

5.1 Sufficient Conditions for $\kappa \geq 2$

In Theorem 5.1, we show that a graph with even girth g, $D \leq g-1$ and minimum degree $\delta \geq 2$ has vertex connectivity $\kappa \geq 2$. The fact that this is not true for graphs with odd girth is demonstrated by the graph obtained by "joining" two Petersen graphs in such a manner that they "share" one vertex, as shown in Figure 5.1, the resulting graph has minimum degree $\delta = 3$, diameter $D = 4$, girth $g = 5$ and vertex connectivity $\kappa = 1$.

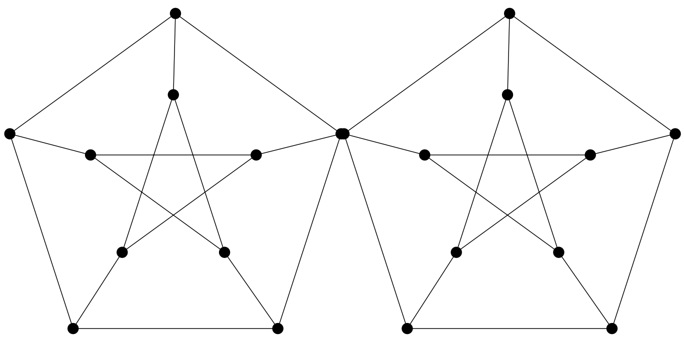


Figure 5.1: A graph with $\delta \geq 2$, $\Delta > 2\delta - 1$, $D \leq g-1$ and $\kappa = 1$.

Nonetheless we improve upon above mentioned known results for graphs with odd girth by showing that $\kappa \geq 2$, for any graph G with odd girth g, diameter at most $D \leq g-1$, and maximum degree $\Delta \leq 2\delta - 1$. The fact that these bounds on the maximum and minimum degree are necessary is demonstrated by the graph in Figure 5.1 having $\Delta = 6 > 2\delta - 1 = 5$ and $\kappa = 1$.

In what follows the notation $[X, C]$ denotes the edges between two sets of vertices X and C, where $X, C \subset V(G)$ and $[x, C]$ the set of edges between a vertex x and the set of vertices C.

$\diamondsuit$ **Theorem 5.1** *Let G be a graph with minimum degree $\delta \geq 2$, maximum degree Δ, girth g and diameter $D \leq g-1$. Then,*

(i) $\kappa \geq 2$, *for g even.*
(ii) $\kappa \geq 2$, *for g odd and $\Delta \leq 2\delta - 1$.*

Proof.

(i) Assume that $\kappa = 1$ and $\{x\}$ is a vertex cut set of G. Let C and C' be two components of $G - x$. Since g is even there exists a vertex $u \in V(C)$ such that $d(u, x) \geq g/2$, for every component C of $G - x$ otherwise G contains a cycle of length $g - 1$. Select two vertices $u \in V(C)$ and $u' \in V(C')$ such that $d(u, x) \geq g/2$ and $d(u', x) \geq g/2$. This principle is demonstrated in Figure 5.2 by a graph with girth $g = 6$ and diameter $D \geq d(u, x) + d(u, v') \geq g$, contradicting the assumption that $D \leq g - 1$. Therefore, $\kappa \geq 2$.

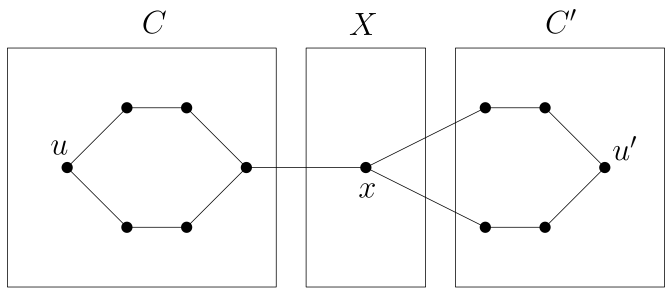


Figure 5.2: Illustration of proof of Theorem 5.1 (i) for a graph with girth 6. $D \geq d(u, x) + d(u', x) \geq g/2 + g/2 = g$.

(ii) Assume that $\kappa = 1$ and $\{x\}$ is a vertex cut set of G. Since g is odd and $D \leq g - 1$ we can apply Theorem 3.5 to establish that edge connectivity $\lambda = \delta$. Then for each component C of $G - x$ the number of edges with one vertex being x and the other vertex in the component C is at least δ. Furthermore, $\lambda = \delta \leq |[x, C]| = |N(x) \cap V(C)|$. Since $G - x$ has at least two components, say C and C' then $2\delta \leq |N(x) \cap V(C)| + |N(x) \cap V(C')| \leq |N(x)| \leq \Delta$, as demonstrated in Figure 5.3 which contradicts the assumption $\Delta \leq 2\delta - 1$, hence $\kappa \geq 2$. ■

5.2 Sufficient Conditions for $\kappa \geq 5$

In Theorem 5.2 we improve on the aforementioned result of Balbuena, Carmona, Fàbrega and Fiol [11] by establishing conditions on the maximum and minimum degree that ensure $\kappa \geq 5$ for any graph with even girth, diameter $D \leq g - 2$ and $\delta \geq 5$. To do this we require the following two lemmas.

Lemma 5.1 *[12, 58, 62, 80, 117] Let G be a graph with girth g, and minimum degree δ. Assume that X is a cut set with cardinality $|X| \leq \delta - 1$. Then, for any connected component C in $G - X$, there exists some vertex $u \in V(C)$ such that $d(u, X) \geq \lfloor (g - 1)/2 \rfloor$.*

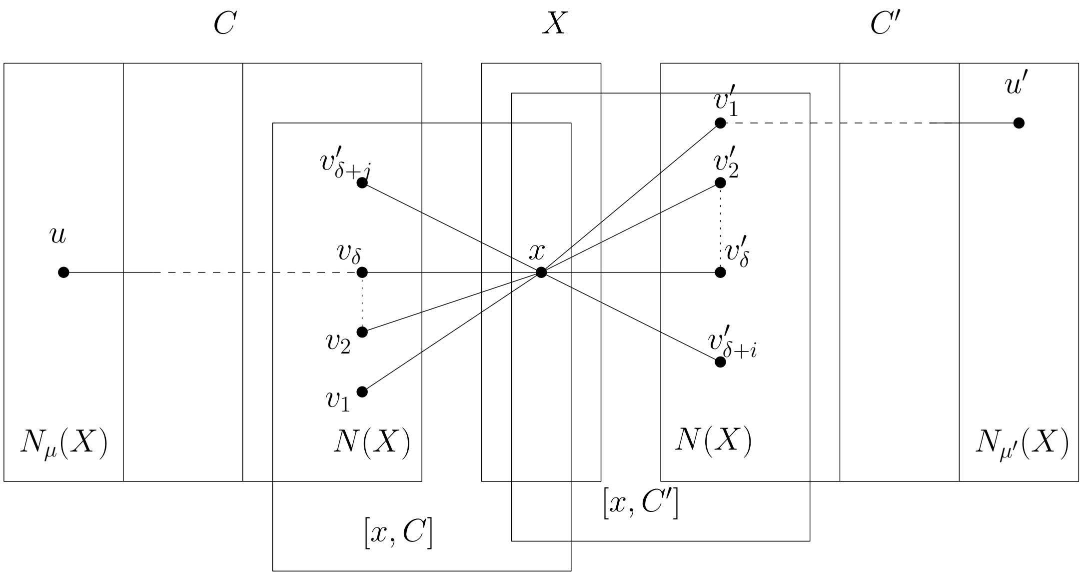


Figure 5.3: Illustration of proof of Theorem 5.1 (ii). $2\delta \leq |N(x) \cap V(C)| + |N(x) \cap V(C')| = \delta + i + \delta + j \leq |N(x)| \leq \Delta$.

Our new Lemma 5.2 is a generalisation of Lemma 2 [18], including non regular graphs with minimum degree $\delta \geq 2$. We consider a graph G with cut set X and a component C of $G - X$, where the maximum distance from any vertex u in C is $\mu = \lfloor (g-1)/2 \rfloor$. We find any vertex u at the maximum distance from X in the component C, has at least $d(u) - |N\mu(u) \cap X|$ neighbours in the component C that are also at the maximum distance from X, more formally,

$\diamondsuit$ **Lemma 5.2** *Let G be a graph with girth g and minimum degree $\delta \geq 2$. Assume that X is a cut set of G and let C be a component of $G - X$ with $\mu = \max\{d(u, X) : u \in V(C)\} = \lfloor (g-1)/2 \rfloor$. Then, for all $u \in N_\mu(X) \cap V(C)$,*

$$|N(u) \cap N_\mu(X) \cap V(C)| \geq d(u) - |N_\mu(u) \cap X|.$$

Proof. Assume $\mu = 1$. Then, since μ is the maximum distance of any vertex $u \in C$ from the cut set X there are no vertices at distance $\mu + 1$ from X. Therefore all vertices in C are neighbours of X, that is, $V(C) \subset N(X)$. Furthermore, all neighbours of u are either in X or at distance $\mu = 1$ from X. Therefore, $|N(u) \cap N(X) \cap V(C)| = d(u) - |N(u) \cap X|$, for all $u \in N(X) \cap V(C)$.

Assume $\mu \geq 2$. Then $N(u) = (N_\mu(X) \cup N_{\mu-1}(X)) \cap V(C)$, for all $u \in N_\mu(X) \cap V(C)$ and $d(u) = |N(u)| = |N(u) \cap N_\mu(X)| + |N(u) \cap N_{\mu-1}(X)|$. Assume $|N(u) \cap N_\mu(X)| < d(u) - |N_\mu(u) \cap X|$, then $|N(u) \cap N_{\mu-1}(X)| > |N_\mu(u) \cap X|$. In this case, by pigeonhole principle, there are two different vertices $z_1, z_2 \in N_{\mu-1}(X) \cap N(u)$ and a vertex $x \in N_\mu(v) \cap X$, such that $d(z_1, X) = d(z_1, x) = \mu - 1$ and $d(z_2, X) = d(z_2, x) = \mu - 1$. Since $d(u, X) = \mu(C) = \mu$, then neither the shortest

(z_1, x)-path nor the shortest (z_2, x)-path contain u. These two paths together with the path of length two z_1, u, z_2, define a cycle whose length is $d(z_1, x)+d(z_2, x)+2 = 2\mu = 2\lfloor(g-1)/2\rfloor < g$, which is a contradiction. Therefore $|N(u) \cap N_\mu(X)| \geq d(u) - |N_\mu(u) \cap X|$. ■

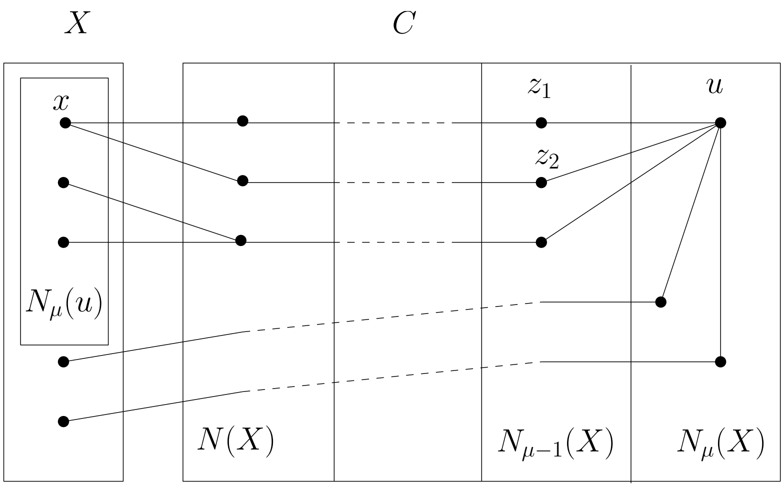


Figure 5.4: Illustration of proof of Lemma 5.2.

We now apply Lemmas 5.1 and 5.2 to prove Theorem 5.2.

♢ **Theorem 5.2** *Let G be a graph with even girth g, minimum degree δ, maximum degree $\Delta \leq 2\delta - 5$, and diameter $D \leq g - 2$. Then $\kappa \geq 5$.*

Proof. Observe that the condition $\delta \leq \Delta \leq 2\delta - 5$ ensures $\delta \geq 5$. Assume $\kappa < 5$. Then by Theorem 3.6(v), we know $\kappa \geq 4$, hence we only need to consider the possibility of $\kappa = 4$. Let $X \subset V(G)$ be a minimum cut set of G. Then, since $|X| = \kappa = 4 \leq \delta - 1$, we can apply Lemma 5.1 and assert that $\mu(C) = \max\{d(u, X) : u \in V(C)\} \geq (g-2)/2$, for all components C of $G - X$. Consider two different, arbitrary components C and C' of $G - X$ and observe that

$$g - 2 \geq D \geq \mu(C) + \mu(C') \geq g - 2,$$

hence every inequality is an equality. Therefore, $D = g - 2$ and $\mu(C) = \mu(C') = (g-2)/2 = \mu$ and Lemma 5.2 applies for both components C and C', that is, for all $u \in N_\mu(X) \cap V(C)$,

$$\begin{aligned} |N(u) \cap N_\mu(X) \cap V(C)| &\geq d(u) - |N_\mu(u) \cap X| \\ &\geq d(u) - |X| \\ &= d(u) - 4 \geq 1. \end{aligned} \tag{5.1}$$

Then u has some neighbour $z \in N_\mu(X) \cap V(C)$, moreover, $N_\mu(u) \cap N_\mu(z) \cap X = \emptyset$ otherwise a cycle of length at most $d(u,x) + d(z,x) + d(u,z) = 2\mu + 1 = g - 1$ is formed.

Let C and C' be two different components of $G - X$. Consider a vertex in each component that is at the maximum distance from the cut set X, namely, $u \in N_\mu(X) \cap C$ and $u' \in N_\mu(X) \cap C'$. Then, since $D = g - 2$, these two vertices must have at least one common neighbour in their neighbourhood at distance μ which are in the cut set X. More formally, $|N_\mu(u) \cap N_\mu(u') \cap X| \geq 1$. Suppose that the neighbourhood of u at distance μ from u in the cut set X is a subset of the neighbourhood of u' at distance μ from u' in the cut set X, that is, $N_\mu(u') \cap X \subset N_\mu(u) \cap X$. Consider a vertex $z \in N(u) \cap N_\mu(X) \cap V(C)$. Since $N_\mu(u) \cap N_\mu(z) \cap X = \emptyset$, it follows that $N_\mu(u') \cap N_\mu(z) \cap X = \emptyset$. A contradiction. Therefore, we have shown that there exists a vertex $x \in N_\mu(u) \cap N_\mu(u') \cap X$ and another $x' \in N_\mu(u) \cap (X - N_\mu(u')) \cap X$, for all $u \in N_\mu(X) \cap C$ and all $u' \in N_\mu(X) \cap C'$. As a result every vertex that is at distance μ from X has at least two neighbours at distance μ in X. More formally, $|N_\mu(u) \cap X| \geq 2$ and $|N_\mu(u') \cap X| \geq 2$, for all $u \in N_\mu(X) \cap V(C)$ and, for all $u' \in N_\mu(X) \cap V(C')$ (see Figure 5.5).

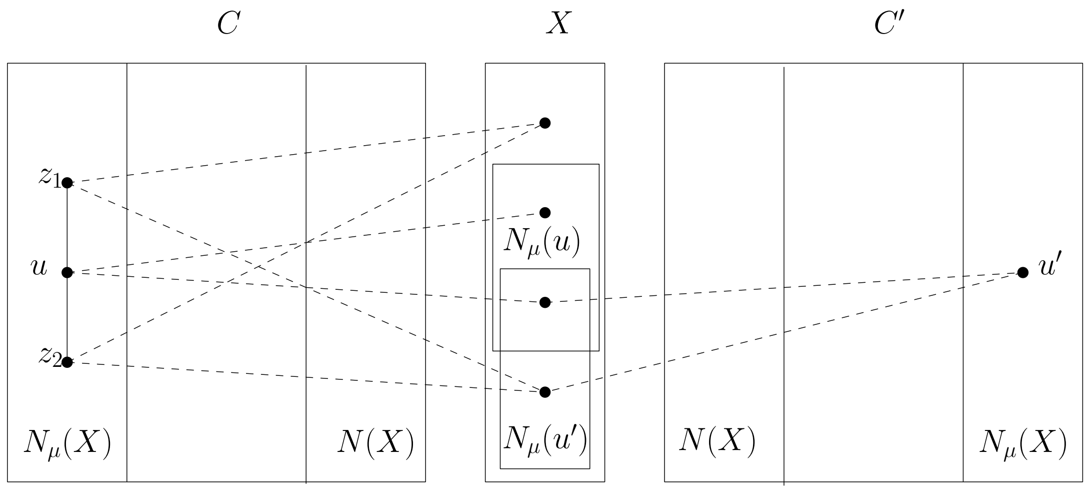


Figure 5.5: Illustration of proof of Theorem 5.2.

Given $u \in N_\mu(X) \cap V(C)$ by Equation 5.2 u has at least one neighbour that is also at distance μ from X. Combining the fact that $|N_\mu(u) \cap X| \geq 2$, for all $u \in N_\mu(X) \cap V(C)$ and $|X| = 4$ we can conclude that $|N_\mu(u) \cap X| = 2$, for all $u \in N_\mu(X) \cap V(C)$. Then by application of Lemma 5.2,

$$
\begin{aligned}
|N(u) \cap N_\mu(X) \cap V(C)| &\geq d(u) - |N_\mu(u) \cap X| \\
&\geq d(u) - 2 \\
&\geq \delta - 2.
\end{aligned}
\tag{5.2}
$$

Therefore, any two shortest paths composed of distinct vertices in $N(u) \cap N_\mu(X) \cap V(C)$ and any vertex in $X \setminus (N_\mu(u) \cap X)$ cannot share any vertex in $N(X \setminus (N_\mu(u) \cap X)) \cap V(C)$ otherwise a cycle of length at most $g-2$ is formed. Then, $|N(x)\cap V(C)| \geq \delta-2$ for each $x \in X\setminus(N_\mu(u)\cap X)$. Applying the same reasoning to the neighbours in $N_\mu(X) \cap V(C)$ of some vertex z which is a neighbour of u, we get $|N(x)\cap V(C)| \geq \delta - 2$, for every $x \in X$. Analogously for any other component $C' \neq C$ of $G-X$; $|N(x)\cap V(C')| \geq \delta-2$, for every $x \in X$. Therefore, $d(x) \geq 2(\delta-2)$ for each $x \in X$, which contradicts our assumption that $\Delta \leq 2\delta-5$. Thus the proposition holds. ■

In this chapter, we have shown that any graph G is 2-connected if diameter $D \leq g-1$ for even girth g, and for odd girth g and maximum degree $\Delta \leq 2\delta-1$, where δ is the minimum degree. Furthermore, we proved that any graph G of diameter $D \leq g-2$ is 5-connected for even girth g and $\Delta \leq 2\delta-1$. In the next chapter we improve on the known results on the superconnectivity of a graph by proving that an r-regular graph G with $r \geq 3$ and diameter at most $g-2$ is super-κ, when g odd. We further extend these new results to include non regular graphs. More particularly, a graph with odd girth g and diameter $D \leq g-2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2-1$ is super-κ.

Proof is an idol before whom the pure mathematician tortures himself.

Arthur Stanley Eddington, The Nature of the Physical World

6

Superconnectivity

In this chapter, we present our results on the superconnectivity, more precisely, vertex superconnectivity, of a graph. Recalling from Chapter 3, a graph is *superconnected*, for short *super-κ*, if all minimum vertex cut sets are trivial. As stated in Section 3.3, Fàbrega and Fiol [58] determined sufficient conditions, in terms of girth g and diameter D, for a graph G to be super-κ or super-λ connected, namely,

$$G \text{ is super-}\lambda \text{ if } \begin{cases} D \leq g-2, & g \text{ odd}, \\ D \leq g-3, & g \text{ even}. \end{cases}$$

$$G \text{ is super-}\kappa \text{ if } \begin{cases} D \leq g-3, & g \text{ odd}, \\ D \leq g-4, & g \text{ even}. \end{cases}$$

In this chapter, we improve upon the above vertex superconnectivity results for r-regular graphs with odd girth g by proving that an r-regular graph G with $r \geq 3$ and diameter $D \leq g-2$ is super-κ, when g odd. These results are detailed in Section 6.1. In Section 6.2, we extend our new results for regular graphs by showing that non regular graphs with odd girth g and diameter $D \leq g-2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2-1$ are super-κ.

6.1 Superconnectivity of Regular Graphs

In this section we present our new results on the superconnectivity of regular graphs. In Theorem 6.1 we prove that an r-regular graph G with $r \geq 3$ and diameter $D \leq g-2$ is super-κ, when g odd. In our proof of Theorem 6.1 we use the known result contained in Proposition 6.1 and our new Lemma 6.1.

Proposition 6.1 *[13] Let $G = (V, E)$ be a connected graph with girth g and minimum degree $\delta \geq 2$. Let $X \subset V$ be a nontrivial vertex cut (κ_1-cut) with cardinality $|X| < \xi(G)$. Then for each connected component C of $G - X$ there exists some vertex $u \in V(C)$ such that $d(u, X) \geq \lceil (g-3)/2 \rceil$; furthermore if g is odd, then $|N_{(g-3)/2}(u) \cap X| \leq 1$.*

Since we are considering graphs with odd girth we use Proposition 6.1 to assert that for each connected component of C of $G - X$ there exists some vertex u in C such that the distance from the vertex u to the cut set X is at least $d(u, X) \geq (g-3)/2$ and there is at most one vertex x in the cut set X such that $d(u, x) = (g-3)/2$, alternatively, $|N_{(g-3)/2}(u) \cap X| \leq 1$. We now use this result to prove some structural properties of a component C, when g is odd and the maximum distance from any vertex u in the component C is equal to $\mu = \max\{d(u, X) : u \in V(C)\} = (g-3)/2$.

$\diamondsuit$ **Lemma 6.1** *Let G be a κ_1-connected graph with odd girth and minimum degree $\delta \geq 3$. Let X be a nontrivial vertex cut (κ_1-cut) with $|X| = \delta$ and assume that there exists a connected component C of $G - X$ such that $\mu = \max\{d(u, X) : u \in V(C)\} = (g-3)/2$. Then the following assertions hold:*

(i) If $u \in V(C)$ is such that $d(u, X) = \mu$ and $|N_\mu(u) \cap X| = 1$, then $d(u) = \delta$ and u has $\delta - 1$ neighbours z_i, for $i = 1, 2, \ldots \delta - 1$, such that $d(z_i, X) = \mu$ and $|N_\mu(z_i) \cap X| = 1$. for all z_i. Moreover, $|N_{\mu+1}(u) \cap X| = \delta - 1$ and X is a set of independent vertices.

(ii) There exists a $(\delta - 1)$-regular subgraph Γ such that every vertex $w \in V(\Gamma)$ has degree $d_G(w) = \delta$ and is at distance μ from the cut set X, more formally, $d(w, X) = \mu$.

(iii) If $g = 5$ then $|N(X) \cap V(C)| \geq \delta(\delta - 1)$.

(iv) If $g \geq 7$ then $|N(X) \cap V(C)| \geq (\delta - 1)^2 + 2$.

Proof. Notice that $g \geq 5$ since $\mu \geq 1$.

(i) Given a vertex $u \in V(C)$ such that $d(u, X) = \mu$ and $|N_\mu(u) \cap X| = 1$, let x_1 be a vertex in X such that $d(u, X) = d(u, x_1) = \mu$ and let $z_1 \in N(u)$ be such that $d(z_1, x_1) = \mu - 1$. Then every vertex in $N(u) \setminus \{z_1\}$ is located into $N_\mu(X) \cap V(C)$, since otherwise, there are at least two vertices, say z_j and z_k, such that $d(z_j, X) = d(z_k, X) = \mu - 1$ and there exist two paths of length μ from u to x_1, namely, $u, z_j, \ldots, x_1$ and $u, z_k, \ldots, x_1$, which form a cycle of length at most $2\mu = 2(g-3)/2 = g - 3 < g$. Therefore, there are $|N(u) \setminus \{z_1\}| = d(u) - 1$ vertices $z_i \in N(u) \cap N_\mu(X)$.

Moreover, the sets $N_\mu(z_i) \cap X$, where $z_i \in N(u) \setminus \{z_1\}$ and $i = 2, \ldots, d(u)$, are pairwise disjoint (see Figure 6.1), because otherwise, there exist at least two vertices, say z_j and z_k in $N(u) \setminus \{z_1\}$ and a vertex $x_k \in X$ such that the $z_j - x_k$ path and the $z_k - x_k$ path both have length μ. Thus a cycle of length at most $2 + 2\mu = 2 + 2(g-3)/2 < g$ is created through the vertices z_j, u, z_k and x_k. Hence,

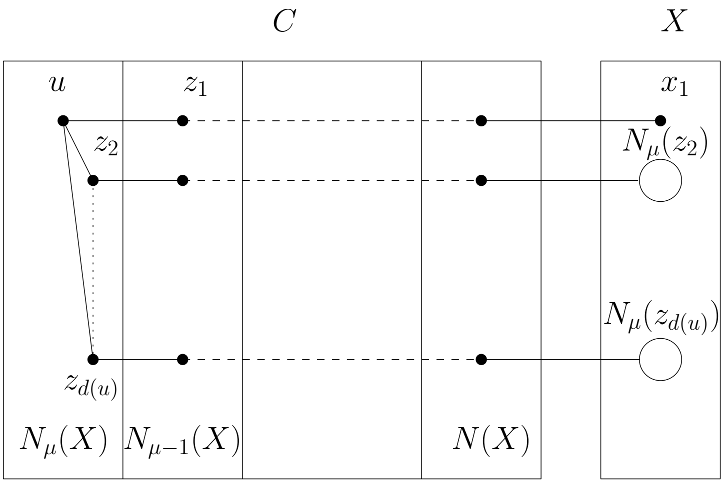


Figure 6.1: The pairwise disjoint sets $N_\mu(z_i) \cap X$.

$$
\begin{aligned}
|X| = \delta &\geq |N_\mu(u) \cap X| + \sum_{i=2}^{d(u)} |N_\mu(z_i) \cap X| \\
&\geq 1 + (d(u) - 1) \\
&= d(u) \geq \delta.
\end{aligned}
\tag{6.1}
$$

From (6.1) the inequalities are forced to be equalities, that is, $d(u) = \delta$, $|N_\mu(z_i) \cap X| = 1$, for every vertex $z_i \in N(u) - z_1$, $i = 2, \ldots, \delta$, and

$$X = (N_\mu(u) \cap X) \cup (\cup_{i=2}^{d(u)} (N_\mu(z_i) \cap X))$$

which means that X is a set of independent vertices. Therefore, we obtain that

$$|N_{\mu+1}(u) \cap X| = |\cup_{i=2}^{d(u)} (N_\mu(z_i) \cap X)| = \sum_{i=2}^{d(u)} |N_\mu(z_i) \cap X| = \delta - 1$$

which finishes the proof of point (i) as shown in Figure 6.2.

(ii) From Proposition 6.1 it follows that there exists a vertex $u \in N_\mu(X) \cap V(C)$ such that $|N_\mu(u) \cap X| = 1$. By item (i) the degree of u is $d(u) = \delta$ and there are $\delta - 1$ vertices $z_i \in N(u) \cap N_\mu(X)$ such that $|N_\mu(z_i) \cap X| = 1$ for $i = 2, \ldots, \delta$. Applying the same reasoning used for proving (i) to the vertices z_i, we obtain $d(z_i) = \delta$, $i = 2, \ldots, \delta$ and each z_i has $\delta - 1$ neighbours $w \in N_\mu(X) \cap V(C)$ such that $|N_\mu(w) \cap X| = 1$. Iterating this reasoning for each of the neighbours of z_i we obtain a $(\delta - 1)$-regular subgraph Γ in $G[N_\mu(X) \cap V(C)]$ such that every $w \in V(\Gamma)$ has $d_G(w) = \delta$.

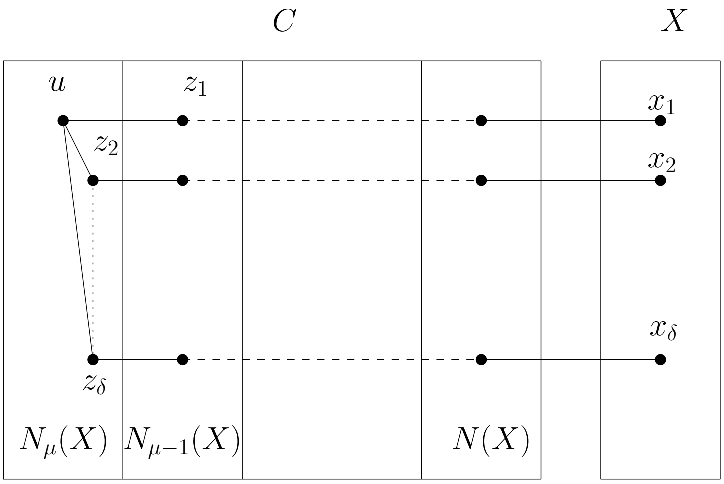


Figure 6.2: $d(u) = \delta$, $|N_{\mu+1}(u) \cap X| = \delta - 1$ and X is a set of independent vertices.

(iii)+(iv) By item (ii) we know that there exists a $(\delta-1)$-regular subgraph Γ in $G[N_\mu(X)\cap V(C)]$ such that every $w \in V(\Gamma)$ has $d_G(w) = \delta$, and by item (i), $|N_\mu(w)\cap X| = 1$, for every $w \in V(\Gamma)$. Let $u \in V(\Gamma)$ be and let $T = (\{u\}\cup N(u)\cup N_2(u))\cap V(\Gamma)$. Then $|N_{\mu-1}(T)\cap N(X)\cap V(C)| \geq |T|$ because otherwise forbidden cycles through u and two different vertices of $N_2(u)\cap V(\Gamma)$ of length at most $2(\mu - 1) + 4 = g - 1$ would be created. Therefore, since $g \geq 5$ we have

$$\begin{aligned} |N_{\mu-1}(T) \cap N(X) \cap V(C)| &\geq |T| \\ &= 1 + (\delta - 1) + (\delta - 1)(\delta - 2) \\ &= 1 + (\delta - 1)^2. \end{aligned} \tag{6.2}$$

Since $u \in V(\Gamma)$ then $d_G(u) = \delta$ which implies that there exists a unique vertex $z_1 \in N(u) \cap N_{\mu-1}(X)$. Let $X = \{x_1, x_2, \ldots, x_\delta\}$ denote the elements of the nontrivial cut set and $N(u)\cap T = \{z_2, \ldots, z_\delta\}$ the neighbours of u included in T. Without loss of generality, let $N_\mu(u)\cap X = \{x_1\}$ and $N_\mu(z_i) \cap X = \{x_i\}$ for $i = 2, \ldots, \delta$. Since $N_\mu(X) = N(X)$ for $g = 5$, we need to consider the two cases $g = 5$ and $g \geq 7$ separately.

Case $g = 5$. Then $\mu = 1$ and ux_1, z_ix_i, $i = 2, \ldots, \delta$ are edges of G. Define the sets $X_i = X \setminus \{x_1, x_i\}$ and $Z_i = N(z_i) \setminus \{u, x_i\}$. We know $|X| = \delta$ and we have shown in (ii) that $d(z_i) = \delta$, therefore $|X_i| = |Z_i| = \delta - 2$. Since $|N(w) \cap X| = 1$, for every $w \in Z_i$, then there exists a perfect matching between each of the sets Z_i and X_i, for all $i = 2, 3, \ldots, \delta$. Let $w^i_k \in Z_i$ denote the $\delta-2$ elements of Z_i such that $w^i_k x_k$, $x_k \in X_i$ are the edges of the matching between Z_i and X_i. Since $d_G(w^i_k) = \delta$ and $\{x_k, z_i\} \subset N(w^i_k)$, then w^i_k must have $\delta - 2$ neighbours more in $N(X)$. Furthermore w^i_k has at most one neighbour w^j_h in $Z_j = N(z_j) \setminus \{u, x_j\}$ for each $j \neq i$, because $g = 5$. Moreover, if w^i_k has a neighbour in Z_k, then there exists an edge $w^i_k w^k_h$ which forms a cycle $w^i_k, x_k, z_k, w^k_h, w^i_k$ of length four, therefore $N(w^i_k) \cap Z_k = \emptyset$ (see Figure 6.3).

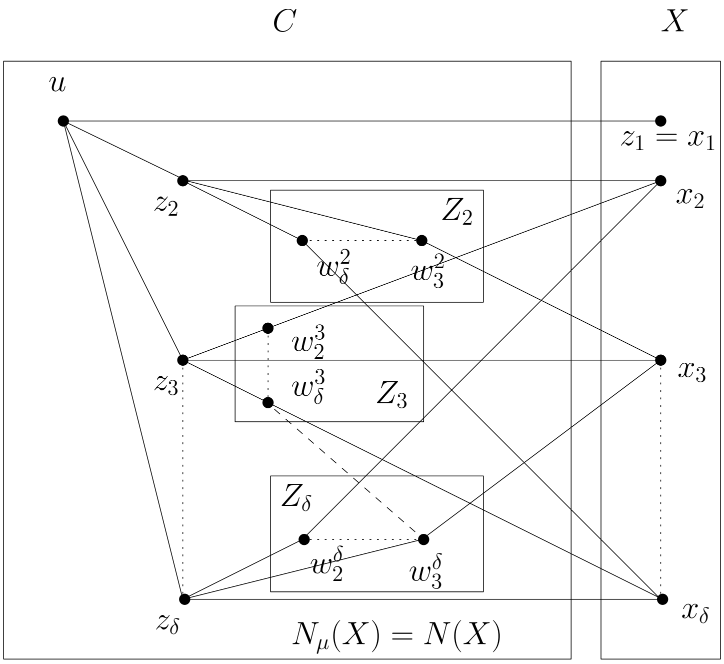


Figure 6.3: An illustration of the proof that $|N(X) \cap V(C)| \geq \delta(\delta - 1)$ for $g = 5$. The dash line is a forbidden edge.

Consequently, $|N(w^i_k) \cap (\cup^\delta_{j=2} Z_j - \{Z_i, Z_k\})| \leq \delta - 3$, which implies that each $w^i_k \in Z_i$ has at least one new neighbour in $N(X) - T$. As an illustration see the graph depicted in Figure 6.6. Therefore,

$$|N(X) \cap V(C)| \geq |T| + |Z_i| \geq 1 + (\delta - 1)^2 + (\delta - 2) = \delta(\delta - 1)$$

and thus (iii) follows.

Case $g \geq 7$. In this case the subgraph of Γ induced by T is a tree and by (6.2) we have $|N(X) \cap V(C)| \geq 1 + (\delta - 1)^2$. We reason by contradiction assuming $|N(X) \cap V(C)| = (\delta - 1)^2 + 1$. Again by (6.2) we know $|N_{\mu-1}(T) \cap N(X) \cap V(C)| = |T| = 1 + (\delta - 1)^2$ which implies $|N_{\mu-1}(u) \cap N(X) \cap V(C)| = 1$, $|N_{\mu-1}(z_i) \cap N(X) \cap V(C)| = 1$ and $|N_\mu(z_i) \cap N(X) \cap V(C)| = \delta - 1$ for $i = 2, \ldots, \delta$. Let $\{z''_1\} = N_{\mu-1}(u) \cap N(X) \cap V(C) = N(x_1) \cap V(C)$. Then, since $g \geq 7$, there exists $w \in N_2(z_i) \cap V(\Gamma)$ for some $i \in \{2, \ldots, \delta\}$ such that $w \notin T$ and $z''_1 \notin N_{\mu-1}(w) \cup N_\mu(w)$, because otherwise a forbidden cycle through u, w, z''_1 of length at most $2\mu + 2$ would be created. Therefore $(N_\mu(w) \cup N_{\mu+1}(w)) \cap X \subseteq X \setminus \{x_1\}$. Applying Lemma 6.1$(i)$ we get $N_{\mu+1}(w) \cap X = \{x_2, \ldots, x_\delta\}$, hence there exists $x_j \in \{x_2, \ldots, x_\delta\}$, $j \neq i$, such that $x_j \in N_\mu(w) \cap N_{\mu+1}(w)$ creating a cycle through x_j and w of length $2\mu + 1$ which is a contradiction. Therefore $|N(X) \cap V(C)| \geq 2 + (\delta - 1)^2$ as required. ■

We now use Lemma 6.1 to prove Theorem 6.1 which improves Theorem 3.5 for regular graphs of odd girth.

$\diamondsuit$ **Theorem 6.1** *Let G be an r-regular graph with $r \geq 3$, odd girth g, and diameter $D \leq g-2$. Then G is super-κ, when $g \geq 5$ and a complete graph otherwise.*

Proof. If $g = 3$ then the diameter is $D \leq g-2 = 1$ and G is a complete graph. Assume $g \geq 5$. We reason by contradiction to show that G is super-κ, for $g \geq 5$. Assume that G is not super-κ. Then there exists a minimum nontrivial cut set X such that $|X| = \kappa_1 = \kappa \leq \delta$. Applying Theorem 3.5 tells us that a graph with odd girth g and diameter $D \leq g-2$ is maximally connected, that is, $\kappa = \delta$. Since by our premises G is an r-regular graph, $\kappa_1 = \kappa = \delta = r$.

Let X be a κ_1-cut with $|X| = r$. Let C and C' denote two components of $G - X$. Let $\mu = \max\{d(u, X) : u \in V(C)\}$ and $\mu' = \max\{d(u', X) : u' \in V(C')\}$ as shown in Figure 6.4. From Proposition 6.1, it follows $\mu \geq (g-3)/2$ and $\mu' \geq (g-3)/2$. Assume $\mu \geq (g-1)/2$ and $\mu' \geq (g-1)/2$. Then the diameter $D \geq \mu + \mu' \geq 2(g-1)/2 = g-1$, contradicting our assumption that $D \leq g-2$. Therefore, there exists at most one component, C', such that $\mu' = (g-1)/2$, and any other component $C \neq C'$ must have $\mu = (g-3)/2$.

Applying Lemma 6.1, we can determine the number of neighbours of X that are in the component C, that is, $|N(X) \cap V(C)| \geq \delta(\delta-1) = r^2 - r$, when $g = 5$ and $|N(X) \cap V(C)| \geq (\delta-1)^2 + 2 = (r-1)^2 + 2$, when $g \geq 7$. Since G is r-regular we can use this result to determine the maximum possible number of neighbours of X in any component $C' \neq C$, namely, $|N(X) \cap V(C')| \leq |N(X)| - |N(X) \cap V(C)| \leq r^2 - (r^2 - r) = r$, when $g = 5$, and $|N(X) \cap V(C')| \leq r^2 - ((r-1)^2 + 2) \leq 2r - 3$, when $g \geq 7$.

Let $F' = [X, V(C')]$ denote the set of edges having one vertex in X and the other vertex in $V(C')$. Then F' is an edge cut set of cardinality $|F'| \leq 2r - 3$. Assume that F' is a trivial edge cut set, then $|F'| = r$ and F' contains all the edges incident with some vertex $x_i \in X$ or $v' \in V(C')$ as demonstrated by the dashed lines in Figure 6.4. If $v' \in V(C')$ then the vertex cut set X consists of r vertices that are exactly the neighbours of v', which contradicts our premise that X is a nontrivial vertex cut set. If $x_i \in X$, then $X \setminus \{x_i\}$ is a vertex cut set with cardinality $|X| - 1$, contradicting our premise that X is a minimal vertex cut set. Therefore F' is a nontrivial edge cut and $\lambda_1 \leq |F'| < \xi = 2r - 2$. Applying Theorem 3.10 we know that $\lambda_1 = \xi$ for $D \leq g - 2$. As a consequence $|X| = \kappa_1 > r$ and G is super-κ. ■

The graph depicted in Figure 6.5 shows a non regular graph of minimum degree $\delta = 3$, girth $g = 5$ and diameter $D = 3$ which is not super-κ. Consequently, the property of degree regularity is essential to establish Theorem 6.1.

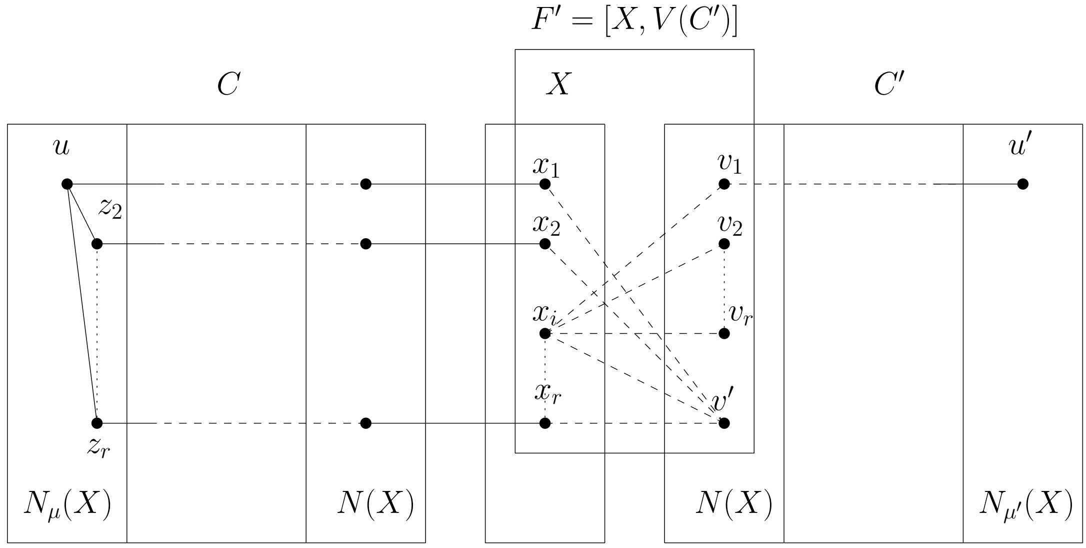


Figure 6.4: Possible trivial edge cut sets F shown in dashed lines.

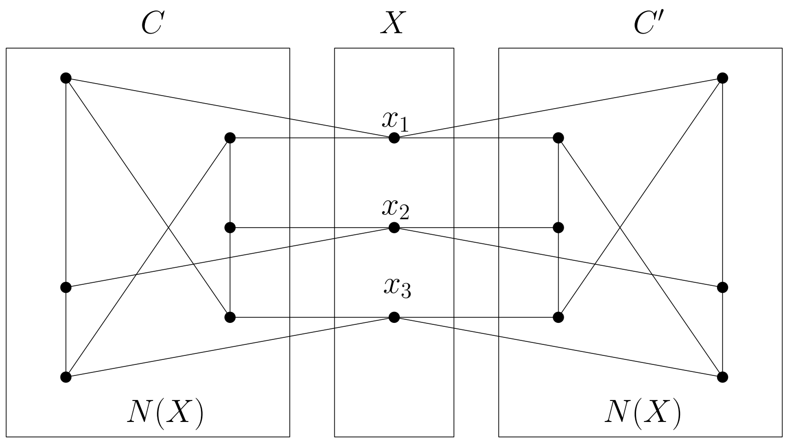


Figure 6.5: A graph with $g = 5$ and $\kappa_1 = \delta = 3$.

In Theorem 6.2 we extend Theorem 6.1 to include non regular graphs with girth g, minimum degree $\delta \geq 3$ and $\Delta \leq 3\delta/2 - 1$. In order to prove Theorem 6.2 the known results in Proposition 6.1 and Lemma 6.1 and two new results contained in Lemmas 6.2 and 6.3 are required.

Lemma 6.2 uses the bounds established in Lemma 6.1 (*iii*) and (*iv*) on the number of vertices in a component C that are also in the neighbourhood of X, that is, $|N(X) \cap |N(C)|$, to establish bounds on the number of edges. Observing that the number of edges $|[X, V(C)]| \geq \delta(\delta - 1)$ demonstrates that the bound given in Lemma 6.1 (*iii*) for $g = 5$ also holds for $g \geq 7$. Moreover, if $\mu = \max\{d(u, X) : u \in V(C)\}$, where X is a cut set and C is a component of $G - X$, we denote the set of vertices in component C at distance μ from X with the notation $N_\mu(X) \cap V(C)$.

$\diamondsuit$ **Lemma 6.2** *Let G be a κ_1-connected graph with odd girth and minimum degree $\delta \geq 3$. Let X be a κ_1-cut with $|X| = \delta$ and assume that there exists a connected component C of $G - X$ such that $\mu = max\{d(u, X) : u \in V(C)\} = (g-3)/2$. Then the following assertions hold.*

(i) For all $x_i \in X$ there exists a vertex $u \in N_\mu(X) \cap V(C)$ such that $d(u, x_i) = (g-3)/2 = \mu$ and $d(u, x_j) = (g-1)/2 = \mu + 1$, for all $x_j \in X - x_i$.

(ii) $|[x, V(C)]| \geq \delta - 1$, for every $x \in X$, and therefore $|[X, V(C)]| \geq \delta(\delta - 1)$.

Proof. Notice that $g \geq 5$ since $\mu \geq 1$.

(i) Suppose that there exists some $x \in X$ such that $d(w, x) \geq \mu + 1$, for all $w \in N_\mu(X) \cap V(C)$. By Lemma 6.1(i), there exists a vertex $u \in N_\mu(X) \cap V(C)$ and $\delta - 1$ vertices $z_2, \ldots, z_\delta \in N_\mu(X) \cap V(C) \cap N(u)$. Then $(\cup_{i=2}^{\delta} N_\mu(z_i) \cup N_\mu(u)) \cap X \subseteq X - x$, which means that there exist two distinct vertices $z_j, z_k \in N(u) \cap N_\mu(X) \cap V(C) : j \neq k$ such that $N_\mu(z_j) \cap N_\mu(z_k) \cap X \neq \emptyset$. However this is a contradiction because cycles of length at most $g - 1$ are created.

(ii) By Lemma 6.1 (i), for every $u \in N_\mu(X) \cap V(C)$ there exists some $x_u \in X$ such that $d(u, x_u) = \mu$ and $d(u, x) = \mu + 1$, for every $x \in X - x_u$. Hence, $|N(u) \setminus (N_\mu(X) \cap V(C))| = 1$.

Let $X = \{x_1, x_2, \ldots, x_\delta\}$ and $u \in N_\mu(X) \cap V(C)$. Suppose that $d(u, x_1) = d(u, X) = \mu$ and let $z_2, \ldots, z_\delta \in N(u) \cap N_\mu(X) \cap V(C)$. Recall that the sets $N_\mu(z_i) \cap X$ must be pairwise disjoint (see Figure 6.1), otherwise G contains a cycle of length $2\mu + 2 = g - 1$ through the vertices $x, \ldots, z_i, u, z_j, \ldots, x$. Suppose that $d(z_i, x_i) = d(z_i, X) = \mu$, $i = 2, \ldots, \delta$. By Lemma 6.1 (i), each vertex z_i has $\delta - 1$ neighbours $u, z_{i1}, \ldots, z_{i\delta-2}$ in $N_\mu(X) \cap V(C)$ and $d(z_{ij}, x_1) = (g-1)/2$, for every $j = 1, \ldots, \delta - 2$, otherwise, as $d(u, x_1) = \mu$ and both u and $z_{ij} \in N_\mu(X) \cap V(C)$ it follows that there exists a cycle of length at most $g-1$. Let $P_{u,v}$ denote a shortest path in G from u to v, hence the number of edges in $P_{u,v}$ is equal to the distance from u to v, that is, $|E(P_{u,v})| = d(u, v)$. Therefore, $|E(P_{z_{ij},x_1})| = \mu + 1$ and $|E(P_{u,x_1})| = \mu$. Also, $V(P_{z_{ij},x_1}) \cap V(P_{z_{ih},x_1}) = \{x_1\}$ and $V(P_{u,x_1}) \cap V(P_{z_{ij},x_1}) = \{x_1\}$, for every distinct $j, h \in \{1, \ldots, \delta - 2\}$, otherwise there exists a cycle of length at most $g - 1$. Then $|[x_1, V(C)]| \geq \delta - 1$.

Furthermore, since $z_i \in N_\mu(X) \cap V(C)$ and $d(z_i, x_i) = \mu$, for $i = 2, \ldots, \delta$, the above result applies for all vertices in X, that is, for every vertex $x_i \in X$ there are at least $\delta - 1$ edges between the cut set and the component C. More formally, $|[x_i, V(C)]| \geq \delta - 1$, for every $i = 1, \ldots, \delta$. Therefore, $|[X, V(C)]| \geq \delta(\delta - 1)$. ■

6.2 Superconnectivity of Non-regular Graphs

In order to prove Lemma 6.3 and Theorem 6.2 we introduce some notation to refer to the sets $S_u^+(v)$, $S_u^=(v)$ and $S_u^-(v)$ which partition the neighbours of v, excluding u, according to their distance from the cut set X. Let $G = (V, E)$ be a graph with vertex cut $X \subset V$, $v \in V \setminus X$, $u \in N(v)$ and $z \in N(v) - u$. Then:

$$\begin{aligned} S_u^+(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) + 1\}; \\ S_u^=(v) &= \{z \in N(v) - u : d(z, X) = d(v, X)\}; \\ S_u^-(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) - 1\}; \end{aligned} \tag{6.3}$$

$\diamondsuit$ **Lemma 6.3** *Let G be a κ_1-connected graph with odd girth $g \geq 5$, minimum degree $\delta \geq 3$ and maximum degree Δ. Let X be a κ_1-cut with δ vertices and assume that there exists a component component C' of $G - X$ such that $\mu' = \mu + 1 = \max\{d(u', X) : u' \in V(C')\} = (g-1)/2$.*

If $D \leq g - 2$ then, for all $u' \in N_{\mu'}(X) \cap V(C')$:

(i) $d(u', x) = (g-1)/2$, *for every* $x \in X$.
(ii) $|[x, N_\mu(u') \cap N(X)]| \geq 1$, *for every* $x \in X$.
(iii) *For every* $x \in X$, $|[x, V(C')]| \leq \Delta - \delta + 1$.
(iv) $|N(u') \setminus \mathcal{F}(C)| \geq 2$ *if* $\Delta \leq 2\delta - 2$.
(v) *For every* $u' \in N_{\mu'}(X) \cap V(C')$, $|N(u') \cap N_{\mu+1}(X) \cap V(C')| \leq \Delta - \delta$.
(vi) *If* $v' \in N(u') \setminus (N_{\mu+1}(X) \cap V(C'))$, *then* $|S_{u'}^+(v')| \leq \Delta - \delta$ *if* $\Delta \leq 2\delta - 2$.
(vii) *There exists some* $v' \in N(u') \setminus (N_{\mu'}(X) \cap V(C'))$ *such that* $|S_{u'}^+(v')| \leq \max\{0, \Delta - \delta - 1\}$ *if* $\Delta < 3\delta/2$.

Proof. Since $D \leq g - 2$, it follows from Proposition 6.1 that for any other component $C \neq C'$ of $G - X$ the maximum distance from the cut set to any other vertex in the component is $\max\{d(u, X) : u \in V(C)\} = (g-3)/2 = \mu$. (As demonstrated in the proof of Theorem 6.1). Recall that for $u \in N_\mu(X) \cap V(C)$ and $z_j, z_k \in N(u) \cap N_\mu(X) \cap V(C)$ the sets $N_\mu(z_i) \cap X$ must be pairwise disjoint (see Figure 6.1), otherwise G contains a cycle of length $2\mu + 2 = g - 1$ through the vertices $x, \ldots, z_i, u, z_j, \ldots, x$.

(i) Since $d(u', X) = (g-1)/2$, then $d(u', x) \geq (g-1)/2$, for all $x \in X$. Suppose that there exist $x_i \in X$ such that $d(u', x_i) \geq (g+1)/2$. By Lemma 6.2 (i) there exists a vertex $u \in N_\mu(X) \cap V(C)$, where $C \neq C'$ is a component of $G - X$, such that $d(u, x_i) = \mu$ and $d(u, x) = \mu + 1$, for all $x \in X - x_i$. Hence,

$$d(u, u') \geq \min\{d(u, x) + d(x, u') : x \in X\} \geq g - 1$$

which contradicts our premise that $D \leq g - 2$.

(ii) Suppose there exists some $x \in X$ such that $|[x, N_\mu(u') \cap N(X)]| = 0$, then $d(x, u') \geq (g+1)/2$ contradicting *(i)*.

(iii) By Lemma 6.2 *(ii)*, we know there are at least $\delta - 1$ edges between a vertex $x \in X$ and a component C, that is, $|[x, V(C)]| \geq \delta - 1$, where $C \neq C'$ is a component of $G - X$. Since $d(x) \leq \Delta$, there are at most $\Delta - (\delta - 1)$ edges incident to x that are adjacent to vertices $v \notin V(C)$. Therefore, $|[x, V(C')]| \leq \Delta - \delta + 1$.

(iv) Assume that $|N(u')\setminus(N_{\mu'}(X)\cap V(C'))| = 1$, then $|N(u)\cap(N_{\mu'}(X)\cap V(C'))| \geq \delta - 1$. By *(ii)*, every $x \in X$ is adjacent to some vertex in $N_\mu(z) \cap N(X)$, for all $z \in N(u') \cap (N_{\mu'}(X) \cap V(C'))$. Then, since for $u \in N_\mu(X) \cap V(C)$ and $z_j, z_k \in N(u) \cap N_\mu(X) \cap V(C)$ the sets $N_\mu(z_i) \cap X$ are pairwise disjoint (see Figure 6.1), it follows that $|[x, V(C')]| \geq \delta$ which means that $d(x) \geq 2\delta - 1$ due to Lemma 6.2 *(ii)*, producing a contradiction. Then $|N(u) \setminus (N_{\mu'}(X) \cap V(C'))| \geq 2$.

(v) Assume that u' has $\Delta - \delta + 1$ neighbours in $N_{\mu'}(X) \cap V(C')$. Then combining the fact that for $u \in N_\mu(X) \cap V(C)$ and $z_j, z_k \in N(u) \cap N_\mu(X) \cap V(C)$ the sets $N_\mu(z_i) \cap X$ are pairwise disjoint (see Figure 6.1), and *(ii)* it follows that $|[x, V(C')]| \geq \Delta - \delta + 2$, for every $x \in X$, contradicting *(iii)*. Hence $|N(u') \cap (N_{\mu'}(X) \cap V(C'))| \leq \Delta - \delta$.

(vi) Assume that the set $S_u^+(v)$ contains s vertices and let $S_u^+(v) = \{u_1, u_2 \ldots, u_s\} \in S_u^+(v)$. Since $v \notin (N_{\mu'}(X) \cap V(C'))$, we have $d(v, X) = d(v, x_v) = \mu$ for some $x_v \in X$. By item *(iv)* we know that $|N(u') \setminus (N_{\mu'}(X) \cap V(C'))| \geq 2$, let $z \in (N(u) - v) \setminus (N_{\mu'}(X) \cap V(C'))$, then $d(z, X) = d(z, x_z) = \mu$ for some $x_z \in X$. Observe that $x_v \neq x_z$ and $d(v, x_z) \geq (g-1)/2$ otherwise a cycle of length at most $g - 1$ exists.

By *(i)*, $d(u_i, x_z) = (g-1)/2$, for all $i = 1, 2, \ldots, s$. Then, $V(P_{u_j,x_z}) \cap V(P_{u_k,x_z}) = \{x_z\}$ for $j \neq k$, otherwise, since $u_j, u_k \in S_u^+(v)$, a cycle of length at most $g - 1$ exists.

Also, since $d(u, x_z) = (g-1)/2$ and $u_i \in N_2(u)$, for all $i = 1, 2, \ldots, s$, it follows that $V(P_{u,x_z}) \cap V(P_{u_i,x_z}) = \{x_z\}$, otherwise a cycle of length at most $g-1$ exists which is a contradiction. Then $|[x_z, V(C')]| \geq s + 1$ yielding $\Delta \geq d(x_z) \geq \delta + s$ because of Lemma 6.2 *(ii)*, hence $s \leq \Delta - \delta$.

(vii) Since $\Delta < 3\delta/2 \leq 2\delta - 2$ we can apply *(vi)* for $v' \in N(u') \setminus (N_{\mu+1}(X) \cap V(C'))$ and assert that $|S_{u'}^+(v')| \leq \Delta - \delta$. If $|S_{u'}^+(v')| < \Delta - \delta$ we are done. Therefore, we assume $|S_u^+(v)| = \Delta - \delta \geq 1$, for every $v \in N(u) \setminus \mathcal{F}(C)$ and let $u_1, \ldots, u_{\Delta-\delta} \in S_u^+(v)$. Clearly $S_u^-(v) \neq \emptyset$.

Assume that $S_u^=(v) \neq \emptyset$ and let $u \in S_u^=(v)$. Then $d(u, X) = d(u, x_u) = (g-3)/2$ for some $x_n \in X$. By item *(i)*, $d(u, x_n) = (g-1)/2$, then $V(P_{n,x_n}) \cap V(P_{u,x_n}) = \{x_n\}$ otherwise a cycle of length at most $g - 2$ exists. As $d(n_i, x_n) = (g-1)/2$, then $V(P_{n,x_n}) \cap V(P_{n_i,x_n}) = \{x_n\}$ and $V(P_{u,x_n}) \cap V(P_{n_i,x_n}) = \{x_n\}$, for all $i = 1, \ldots, \Delta - \delta$ and $d(u, x_n) = (g-1)/2$, otherwise a cycle of length at most $g - 1$ exists. Furthermore, $V(P_{n_i,x_n}) \cap V(P_{n_j,x_n}) = \{x_n\}$, for every distinct $i, j = 1, \ldots, \Delta - \delta$. Then $|[x_n, V(C)]| \geq \Delta - \delta + 2$ which contradicts *(iii)*, therefore $S_u^=(v) = \emptyset$, for every $v \in N(u) \setminus \mathcal{F}(C)$. Hence $d(v) = 1 + |S_u^-(v)| + |S_u^+(v)| = 1 + |S_u^-(v)| + \Delta - \delta$. If $|S_u^-(v)| = 1$ then $d(v) = \Delta - \delta + 2$ following $\Delta \geq 2\delta - 2$ which contradicts our assumption that $\Delta \leq 3\delta/2 - 1$. Therefore, $|S_u^-(v)| \geq 2$.

By (v), $|N(u) \cap \mathcal{F}(C)| \leq \Delta - \delta$. Then there are $d(u) - \Delta + \delta \geq 2\delta - \Delta$ vertices in $N(u) \setminus \mathcal{F}(C)$. Let $v_1, \ldots, v_{2\delta-\Delta}$ in $N(u) \setminus \mathcal{F}(C)$. Since $|S_u^-(v_i)| \geq 2$, for every $i = 1, \ldots, 2\delta - \Delta$, then

$$\delta = |X| \geq \sum_{i=1}^{2\delta-\Delta} |N_{(g-5)/2}(S_u^-(v_i)) \cap X| \geq 2(2\delta - \Delta),$$

clearly $N_{(g-5)/2}(S_u^-(v_i)) \cap N_{(g-5)/2}(S_u^-(v_j)) \cap X = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, 2\delta - \Delta$.

Since $\Delta < 3\delta/2$ we obtain that $|X| \geq 4\delta - 2\Delta \geq \delta + 1$, which is a contradiction. ■

$\diamondsuit$ **Theorem 6.2** *Let G be a graph with odd girth g, diameter $D \leq g - 2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2 - 1$. Then G is super-κ.*

Proof. If $g = 3$ then the diameter is $D \leq g - 2 = 1$ and G is a complete graph, which is by definition superconnected. Assume $g \geq 5$. We reason by contradiction to show that G is super-κ, for $g \geq 5$. Assume that G is not super-κ. Then there exists a minimum nontrivial vertex cut X such that $|X| = \kappa_1 = \kappa \leq \delta$. Applying Theorem 3.5 tells us that a graph with odd girth g and diameter $D \leq g - 2$ is maximally connected, that is, $\kappa = \delta$.

Let X be a κ_1-cut with $|X| = \delta$. Let C and C' denote two components of $G - X$. Let $\mu = \max\{d(u, X) : u \in V(C)\}$ and $\mu' = \max\{d(u', X) : u' \in V(C')\}$ as shown in Figure 6.4. From Proposition 6.1, it follows $\mu \geq (g - 3)/2$ and $\mu' \geq (g - 3)/2$. Assume $\mu \geq (g - 1)/2$ and $\mu' \geq (g - 1)/2$. Then the diameter is

$$D \geq d(u, u') \geq d(u, X) + d(u', X) \geq \mu + \mu' \geq 2(g - 1)/2 = g - 1.$$

This contradicts our assumption that $D \leq g-2$. Therefore, there exists at most one component, C', such that $\mu' = (g - 1)/2$, and any other component $C \neq C'$ must have $\mu = (g - 3)/2$.

Assume that every component of $G - X$ has $\mu = (g - 3)/2$. Then by application of Lemma 6.2 (ii) there are at least $\delta(\delta - 1)$ edges between the vertex cut X and each of the components $C, C' \in G - X$. Furthermore, the sum of edges having at least one vertex $x \in X$ is at most $\Delta|X| = \Delta\delta$. Therefore,

$$\delta\Delta \geq |[X, V(G)]| \geq |[X, C]| + |[X, C']| \geq 2\delta(\delta - 1).$$

Dividing the above inequality by δ gives $\Delta \geq 2\delta - 2$ which contradicts our assumption that $\Delta < 3\delta/2$ for values of $\delta \geq 3$. Therefore, there exists a component, C', such that $\mu' = \max\{d(u', X) : u' \in V(C')\} = (g-1)/2$ and another component C such that $\mu = \max\{d(u, X) : u \in V(C)\} = (g-3)/2$.

Let $X = \{x_1, \ldots, x_\delta\}$ and $u \in N_\mu(X) \cap V(C)$, by Lemma 6.1 (i), there exists some vertex $x_1 \in X$, such that $d(u, X) = d(u, x_1) = (g-3)/2$ and $d(u, x_i) = (g-1)/2$, for every $i = 2, \ldots, \delta$, hence, without lost of generality, assume that $z_2, \ldots, z_\delta \in N(u') \cap N_\mu(X) \cap V(C)$. Observe that $N_\mu(z_i) \cap N_\mu(z_j) \cap X = \emptyset$ otherwise a cycle is formed through z_i, z_j and a vertex $x \in X$. The length of the cycle being,

$$d(x, z_i) + d(x, z_j) + d(z_i, z_j) = \mu + \mu + 2 = (g-3)/2 + (g-3)/2 + 2 = g - 1,$$

resulting in a contradiction. Hence, $d(z_i, x_i) = (g-3)/2$, for every $i = 2, \ldots, \delta$.

Let $u' \in N'_\mu(X) \cap V(C')$, where $\mu' = \max\{d(u', X) : u' \in V(C')\} = (g-1)/2$ then by Lemma 6.3 (v), $|N(u') \cap N'_\mu(X) \cap V(C')| \leq \Delta - \delta$. Suppose that $|N(u') \cap N'_\mu(X) \cap V(C')| = t$, where $t \in \{0, \ldots, \Delta - \delta\}$. Since the sets $N\mu(u) \cap N(X) \cap V(C)$ are pairwise disjoint, for all $u \in N(u) \cap N'_\mu(X) \cap V(C)$, for all $u \in N(u) \cap N'_\mu(X) \cap V(C)$, by Lemma 6.3 (ii) every $x \in X$ satisfies

$$\left| \left[x, \bigcup_{u \in N'_\mu(u) \cap N(X) \cap V(C')} N_\mu(u) \cap N(X) \cap V(C) \right] \right| \geq t.$$

For every $v \in N(u) \setminus (N'_\mu(X) \cap V(C'))$ clearly $|S_u^-(v) \cup S_u^=(v) \cup S_u^+(v)| \geq \delta - 1$; as by Lemma 6.3 (vi), $|S_u^+(v)| \leq \Delta - \delta$, then $|S_u^-(v) \cup S_u^=(v)| \geq 2\delta - \Delta - 1$ and by Lemma 6.3 (vii), there exist a vertex $v^* \in N(u) \setminus (N'_\mu(X) \cap V(C'))$ such that $|S_u^-(v^*) \cup S_u^=(v^*)| \geq 2\delta - \Delta$.

Then, since $g \geq 5$,

$$\left| \bigcup_{v \in N(u) \setminus (N'_\mu(X) \cap V(C'))} (S_u^-(v) \cup S_u^=(v)) \right| \geq (2\delta - \Delta - 1)(\delta - t) + 1.$$

Since, for all $v, v' \in N(u) \setminus (N'_\mu(X) \cap V(C'))$ and $w \in N(u) \cap N'_\mu(X) \cap V(C')$, the sets $N_{(g-5)/2}(S_u^=(v)) \cap N(X)$, $N_{(g-5)/2}(S_u^=(v')) \cap N(X)$, $N_{(g-5)/2}(v) \cap N(X)$, $N_{(g-5)/2}(v') \cap N(X)$ and $N_{(g-3)/2}(w) \cap N(X)$ are pairwise disjoint, then

$$|N(X)| \geq (2\delta - \Delta - 1)(\delta - t) + 1 + t.$$

Thus, by Lemma 6.3 (ii) there exist a vertex $x \in X$ such that

$$\big|[x, V(C)]\big| \geq t + \left\lceil \frac{(2\delta - \Delta - 1)(\delta - t) + 1}{\delta} \right\rceil.$$

Since $\frac{2\delta-\Delta-1}{\delta} \leq 1$, then

$$\big|[x, V(C)]\big| \geq \left\lceil \frac{(2\delta - \Delta - 1)\delta}{\delta} + \frac{1}{\delta} \right\rceil = \lceil 2\delta - \Delta - 1 + \frac{1}{\delta} \rceil$$

Furthermore, by Lemma 6.3 (*iii*), $|[x, V(C)]| \leq \Delta - \delta + 1$, then

$$\lceil 2\delta - \Delta - 1 + \frac{1}{\delta} \rceil \leq \Delta - \delta + 1$$

which allows us to conclude that $\lceil \frac{3\delta}{2} + \frac{1}{2\delta} - 1 \rceil \leq \Delta$, which contradicts the assumption that $\Delta \leq 3\delta/2 - 1$.

Therefore G is super-κ. ■

The fact that Theorem 6.2 is tight is demonstrated in Figure 6.6 which depicts a graph G, having $\delta = 3$ and $\Delta = 4$, odd girth $g = 5$ and diameter $D = g - 2 = 3$ which is not super-κ.

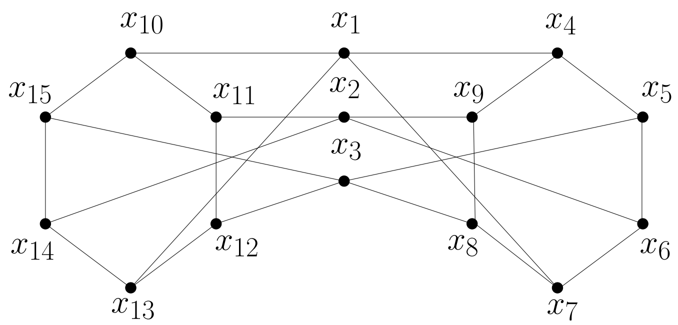


Figure 6.6: A graph with $\delta = 3$, $\Delta = 4$, odd girth $g = 5$ and $D = g - 2 = 3$ having $\kappa_1 = \delta = 3$.

In this chapter we improved upon the results of Fàbrega and Fiol [58] who determined sufficient conditions, in terms of girth g and diameter D, for a graph G to be super-κ or super-λ connected. We showed that an r-regular graph G, with $r \geq 3$, odd girth g and diameter $D \leq g - 2$ is super-κ. Furthermore we proved that a non regular graph with odd girth g and diameter $D \leq g - 2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2 - 1$ is super-κ. These new results are collated, along with those presented in the previous chapter and other known results on sufficient conditions in terms of diameter, girth, maximum and minimum degree, for good connectivity, and presented in the following chapter.

It is impossible to be a mathematician without being a poet in soul.
Sofia Kovalevskaya (1850 - 1891)

7

Summary and Future Research

In this thesis, we considered two important problems in extremal graph theory. Firstly, determining the extremal number, $ex(n;t)$, and finding corresponding extremal graphs $G \in EX(n;t)$, for given values of n and t. Secondly, establishing sufficient conditions, in terms of degree, diameter and girth, under which a graph has good connectivity. In this chapter we summarise the main findings of the thesis and present a number of open problems for future research.

The central problem when considering extremal $\{C_2, C_3, \ldots, C_n\}$-free graphs is that of determining the value of $ex(n;t)$ and finding all the corresponding extremal graphs. There are infinitely many values of n and t for which this problem can be considered. For some values of n and t this problem has been shown to be very difficult, for example, determining if the extremal number $ex(3250, 4)$ is 92,625 or not. This question is equivalent to determining the existence of the Moore graph with degree 57 and diameter 2 which is a question that is central to the degree/diameter problem. The same question can be expressed in terms of the degree/girth problem, more precisely, "Does the (57,5)-cage have order 3250, or not?". The answer to this question has been sought for over 50 years. For this reason we focus our future research on some particular values of $ex(n;t)$ as detailed in the following paragraphs.

No new exact values of $ex(n;4)$ or corresponding extremal graphs have been determined since those published for $n \leq 200$ by Garnick, Kwong and Lazebnik [66] in 1993. However, in Section 4.4, we improved the current best known lower bounds on $ex(n;t)$ for $73 \leq n \leq 200$, for all but nine values of n. Furthermore, we were able to create graphs with order n and girth $t+1$ that had one more edge than the graphs that provide the current best known lower bounds given in [66] for $ex(35;4)$ and $ex(45;4)$. Despite many attempts, we were unable to make any other improvements on $ex(n;4)$, for $31 \leq n \leq 49$ and, therefore, we suspect these lower bounds are tight.

In Chapter 3, we examined the relationship between the degree/girth problem and the problem of determining the extremal number and finding the corresponding extremal graphs. In 1993, Garnick, Kwong and Lazebnik [66] showed that $EX(19;4) = \{(4,5)\text{-cage}\}$ and $ex(19;4) = 38$. Furthermore, the same authors showed that $ex(30;4) = 76$, from which we can deduce that the four (5,5)-cages are not extremal since these cages have 30 vertices and 75 edges. Recently, Abajo and Diánez [3] showed that $EX(24;6) = \{(3,7)\text{-cage}\}$. Moreover, we know that all Moore cages are extremal graphs. Currently the only known cages that are known not to be extremal graphs are the four (5,5)-cages. In Table 7.1 we list all current known cages for which it is currently unknown whether they are extremal graphs or not. We would like to determine if these graphs are extremal or not.

$n(k,g)$	$M(k,g)$	No. of (n,k)-cages	$ex_l(n;t)$	$ex(n;t)$	$ex_u(n;t))$
$n(6,5) = 40$	37	1	120	$\leq ex(40;4) \leq$	125
$n(7,6) = 90$	86	1	315	$\leq ex(90;5) \leq$	322
$n(4,7) = 67$	53	?	134	$\leq ex(67;6) \leq$	144
$n(3,9) = 58$	46	18	87	$\leq ex(58;8) \leq$	91
$n(3,10) = 70$	62	3	105	$\leq ex(70;9) \leq$	107
$n(3,11) = 112$	94	1	168	$\leq ex(112;10) \leq$	173

Table 7.1: Cages that attain the current best known lower bound on $ex(n;t)$, for $n = n(k,g)$ and $t = g-1$.

In Section 4.5, we showed that a number of graphs when subdivided form infinite families of extremal graphs, namely, the complete graphs K_2, K_3, and K_4, the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,4}$, the Petersen graph, the Heawood graph and the Tutte-Coxeter cage. During the course of our research we found some other subdivided graphs that gave the current best known lower bounds on the $ex(n;t)$ for some values of n and t. We suspect that some of these graphs may also form infinite families of extremal graphs when subdivided. In particular, we would like to determine if the graphs listed in Table 7.2 are extremal or not.

Graph	Subdivided graph	$ex_l(n;t)$	$ex(n;t)$	$ex_u(n;t)$
$K_{4,4}$	$s_1K_{4,4}$	32	$ex(24;7) =$	32
$K_{4,4}$	$s_2K_{4,4}$	48	$\leq ex(40;11) \leq$	49
McGee (3,7)-cage	s_1MG	72	$\leq ex(60;13) \leq$	74
McGee (3,7)-cage	s_2MG	108	$\leq ex(96;20) \leq$	110
(3,9)-cage	s_1C39	174	$\leq ex(145;17) \leq$	180
(3,9)-cage	s_2C39	261	$\leq ex(232;26) \leq$	267
(3,10)-cage	s_1C310	210	$\leq ex(175;19) \leq$	214
(3,10)-cage	s_2C310	315	$\leq ex(280;29) \leq$	319
Balaban (3,11)-cage	s_1Bal	336	$\leq ex(280;21) \leq$	344
Balaban (3,11)-cage	s_2Bal	504	$\leq ex(448;32) \leq$	512
Benson (3,12)-cage	s_1Ben	378	$\leq ex(315;23) \leq$	381
Benson (3,12)-cage	s_2Ben	567	$\leq ex(504;35) \leq$	570

Table 7.2: Subdivided graphs that provide the current best known lower bound on the corresponding extremal numbers.

In Chapter 4, we introduced our "Grow and Prune" (GAP) algorithm. The pseudocode for our GAP algorithm is contained in Appendix A. We used our GAP algorithm to generate a number of graphs with size greater than the current best known lower bounds on $ex(n;t)$, for $t = 4, 5, \ldots, 11$ and $n \leq 200$. Notably, for $ex(n;4)$ many of our new lower bounds improved upon the current best known lower bounds established by Garnick, Kwong and Lazebnik [66]. Furthermore, for $t = 4$ and 6, we were able to improve upon the lower bounds recently published by Abajo and Diánez [2] and Abajo, Balbuena and Diánez [1].

Our GAP algorithm "Grows" a graph G by finding two arbitrary vertices u and v whose distance from each other is equal to the diameter of G, that is, $d(u,v) = D$. Then, if $D = t$ we add the edge $\{u,v\}$, otherwise we add the path $\{\{u,n+1\},\{n+1,n+2\},\ldots,\{n+k,v\}\}$, where $n+1, n+2, \ldots, n+k$ are all new vertices in G. At the end of each iteration of the Grow algorithm we have either a graph G' with $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)|+1$ or a graph G'' with $|V(G'')| = |V(G)|+k$ and $|E(G')| = |E(G)|+k+1$.

We believe that there might be a more productive way to chose the vertices u and v that may improve the growing algorithm. Furthermore, we have experimented with growing a graph by subdividing an arbitrary edge on a girth cycle but the lower bounds generated using the subdivision method were equal to, or inferior to, those given by our current growing algorithm. However, if the goal was to grow an extremal network, vertex by vertex, in such a manner that the connectivity is at least 2, then subdivision would be the superior method of growth.

Our GAP algorithm "Prunes" a graph G in two steps. The first step consists of finding g vertices that lie on a girth cycle of G and deleting them one by one. The second step consists of finding a vertex $v \in G$, such that, $deg(v) = \delta$ and deleting it. The second step is then iterated until $|V(G)| = t+1$. In some cases we ran the second step of the algorithm manually and improved the results by carefully selecting the next vertices to be deleted, for example, after deleting all of the vertices that were on a girth cycle $C_g \subset G$ we would then delete all vertices in another cycle $C_k \in G$ such that $|V(C_k) \cap V(C_g)|$ is maximal. In the future we would like to automate this process.

Our GAP algorithm was designed to be used given regular graphs as input graphs. This decision was made due to the availability of dense (k,g)-graphs that have been found by researchers working on the degree/girth problem. In Section 4.5, we found new families of extremal graphs that are not degree regular and intend to revise the GAP algorithm to take advantage of these graphs in the future.

The current version of our GAP algorithm takes an integer t and an array of integer which are lower bounds on $ex(n;t)$, for $n = 1, 2, \ldots 200$ as input. Changing the input to a matrix of lower bounds with dimensions n and t would enable us to perform other operations on input graphs, for example subdivision. Furthermore, by maintaining a matrix of upper bounds we could measure the size of the gap between the upper bound and the current best known lower bound for particular values of $ex(n;t)$. In the context of the degree/diameter problem, a graph of order $M_{\Delta,D} - d$ is said to have *defect* d. Similarly, when considering the degree/girth problem a graph of order $M(k,g) + e$ is said to have *excess* e. A similar measure for the size of the extremal graphs $EX(n;t)$ could be the *edge defect*, where the defect is measured in relation to the upper bound given by application of the Moore bound on irregular graphs, $M(\bar{k},g)$. It would be interesting to determine how many extremal graphs attain the upper bound given by the Moore bound for irregular graphs.

In Appendix B, we provide a summary of the current best known lower bounds on $ex(n;t)$, for $t = 4, 5, \ldots, 11$ and $n \leq 200$ and the upper bounds generated by application of the Moore bound for irregular graphs. When the extremal number is known it is shown in bold font in the corresponding table cell. In order to present an intelligible summary we have not acknowledged the source of individual lower bounds on $ex(n;t)$. References to the source of the current best known lower bounds on $ex(n;t)$ can be found in Chapter 4.

In addition to determining the extremal graphs and extremal numbers for the subdivided graphs detailed in Section 4.5, we used the lower bounds on the extremal numbers generated by our GAP algorithm and a number of structural properties of extremal graphs to establish further extremal numbers that were previously unknown. In particular, $ex(n;6)$, for $n = 30, 31, 32$; $ex(n;8)$, for $n = 23, 24, 25, 26$; $ex(n;9)$, for $n = 26, 27, 28, 29$; and $ex(127;11)$.

In Chapter 5, we improved upon a result by Balbuena and Marcote [18] by showing that any graph G is 2-connected if diameter $D \leq g-1$ for even girth g, and for odd girth g and maximum degree $\Delta \leq 2\delta - 1$, where δ is the minimum degree. Furthermore, we extended the results of Balbuena, Carmona, Fàbrega and Fiol [11], by proving that any graph G of diameter $D \leq g-2$ is 5-connected for even girth g and $\Delta \leq 2\delta - 1$.

In Chapter 6, we improved known results by Fàbrega and Fiol [58] on the superconnectivity of a graph. We proved that an r-regular graph with odd girth g, $r \geq 3$ and diameter $D \leq g-2$ is super-κ. We then extended these results by showing that non regular graphs with odd girth g and diameter $D \leq g-2$, minimum degree $\delta \geq 3$ and maximum degree $\Delta \leq 3\delta/2 - 1$ are super-κ.

Table 7.3 contains a summary of current known sufficient conditions to ensure good connectivities in terms of diameter, girth, maximum, and minimum degree.

Our investigation of extremal graphs was motivated by the belief that they have a number of properties that are desirable in networks. In particular, we believe that extremal graphs have good connectivity properties. In Chapter 3, we compared the known results on the connectivity of Moore graphs and cages in order to provide a benchmark. In summary, all Moore graphs are maximally connected. Fu, Huang and Rodger [64] conjectured that every (k,g)-cage is k-connected. We gave a number of results that support this conjecture. Since $D \leq t-1 \leq g-2$, for all extremal graphs, applying Theorem 3.8 tells us that the extremal graphs are maximally connected, that is, $\kappa = \lambda = \delta$ for even t. In summary, all extremal graphs having minimum

D and g	Connectivity	Conditions		Reference
$D \leq g+1$	$\kappa \geq 2$	g odd	if G is r-regular, $r \leq 3$	[18]
	$\lambda \geq 2$	g odd		
$D \leq g$	$\kappa \geq 2$	g odd	if G is r-regular	[18]
	$\kappa \geq 2$	g even	if G is r-regular, $r \leq 3$	
	$\lambda \geq 2$	g even		
$D \leq g-1$	$\kappa \geq 2$	g odd	if $\Delta \leq 2\delta - 1$	$\diamondsuit$
	$\kappa \geq 2$	g even		$\diamondsuit$
	$\kappa \geq \min\{r, 3\}$	g odd	if G is r-regular	[18]
	$\lambda = \delta$	g odd		[117]
	$\lambda \geq \min\ \{\delta, 4\}$	g even		[11]
$D \leq g-2$	$\kappa = \delta$	g odd		[117]
	$\kappa \geq \min\{\delta, 4\}$	g even		[11]
	$\kappa \geq 5$	g even	if $\Delta \leq 2\delta - 5$	$\diamondsuit$
	$\kappa \geq \min\ \{r, 6\}$	g even	if G is r-regular	[18]
	super-κ	g odd	if $\Delta \leq 3\delta/2 - 1$ for $\delta \geq 3$	$\diamondsuit$
	super-λ	g odd		[58]
$D \leq g-3$	super-κ	g odd		[58]
	super-λ			
$D \leq g-4$	super-κ			[58]

Table 7.3: Sufficient conditions to ensure good connectivities in terms of diameter, girth, maximum and minimum degree.

degree $\delta \geq 2$ and odd girth are super edge connected and maximally connected when t is even. Tang, Lin, Balbuena and Miller [118] conjectured that extremal graphs are also maximally connected for odd t. We would like to solve this conjecture and believe that in doing so we would provide some insight to the conjecture that every (k, g)-cage is k-connected.

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A

Grow and Prune Algorithm

Algorithm 1 Grow and Prune

```
Input: integer t
  integer LowerBounds[200]
  graph G
Output: LowerBounds[200]
  procedure grow(G,t,LowerBounds)
  begin
  repeat
    n:=NumberOfVertices(G)
    d:=Diameter(G)
    k:=t-d
    Find two vertices u and v such that d(u,v)=d
    if k==0 then
      G:=AddEdge(G,{u,v})
    else
      G:=AddVertices(G, [n+1, ..., n+k])
      G:=AddEdges(G, {{u,n+1},{n+1,n+2},..., {n+k,v}})
    end if
    if LowerBounds[NumberOfVertices(G)] < NumberOfEdges(G) then
      LowerBounds[NumberOfVertices(G)] := NumberOfEdges(G)
    end if
  until NumberOfVertices(G)>200
  end procedure

  procedure prune(G,t,LowerBounds)
  begin
  g:=girth(G)
  GirthVertex[g]:=GirthCycle(G)
  for i = 1 → g do
    G:=DeleteVertex(G,GirthVertex[i])
    if LowerBounds[NumberOfVertices(G)] < NumberOfEdges(G) then
      LowerBounds[NumberOfVertices(G)] := NumberOfEdges(G)
    end if
  end for
  repeat
    v:=GetMinimumDegreeVertex(G)
    G:=DeleteVertex(G,v)
    if LowerBounds[NumberOfVertices(G)] < NumberOfEdges(G) then
      LowerBounds[NumberOfVertices(G)] := NumberOfEdges(G)
    end if
  until NumberOfVertices(G)=t+1
  end procedure
```

B

Summary of known values of $ex(n;t)$, for $t = 4, 5, \ldots, 11$ and $n \leq 200$

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**5**	**6**	**8**	**10**	**12**
10	**15**	**16**	**18**	**21**	**23**	**26**	**28**	**31**	**34**	**38**
20	**41**	**44**	**47**	**50**	**54**	**57**	**61**	**65**	**68**	**72**
30	**76**	80-81	85-86	87-91	90-96	95-101	99-106	104-111	109-115	114-120
40	120-125	124-129	129-134	134-139	139-144	145-149	150-154	156-159	162-164	168-169
50	**175**	176-180	178-185	181-191	185-196	188-202	192-207	195-213	199-218	203-224
60	207-230	212-236	216-242	221-248	226-253	231-260	235-266	240-272	245-278	250-284
70	255-290	260-297	265-303	271-309	278-316	285-322	291-329	298-335	305-342	312-348
80	320-355	322-362	327-369	334-375	341-382	348-389	355-396	362-403	369-410	376-417
90	384-424	392-431	399-438	407-446	415-453	423-460	432-467	436-475	438-482	440-490
100	443-497	445-505	447-512	450-520	452-527	458-535	465-543	472-550	480-558	488-566
110	496-574	504-582	512-589	520-597	528-605	536-613	544-621	552-630	560-638	568-646
120	576-654	585-662	593-671	602-679	611-687	620-695	630-704	634-712	638-721	641-729
130	644-738	647-746	650-755	653-764	657-772	666-781	674-790	683-798	692-807	700-816
140	709-825	717-834	726-843	735-852	744-860	753-870	762-879	771-888	780-897	789-906
150	798-915	808-924	817-933	827-943	837-952	847-961	858-971	862-980	865-989	868-999
160	871-1008	873-1018	875-1027	878-1037	880-1046	883-1056	886-1066	892-1075	901-1085	910-1095
170	920-1105	930-1114	932-1124	935-1134	938-1144	941-1154	949-1164	958-1174	968-1184	977-1194
180	986-1204	995-1214	1004-1224	1013-1234	1022-1244	1032-1254	1042-1264	1052-1275	1062-1285	1072-1295
190	1082-1306	1092-1316	1102-1326	1112-1337	1122-1347	1132-1358	1142-1368	1152-1379	1163-1389	1173-1400
200	1184-1410									

Table B.1: Summary of known values $ex(n;4)$, $ex_l(n;4)$ and $ex_u(n;4)$, for $n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**6**	**7**	**9**	**10**
10	**12**	**14**	**16**	**18**	**21**	**22**	**24**	**26**	**29**	**31**
20	**34**	**36**	**39**	**42**	**45**	**48**	**52**	**53**	**56**	**58**
30	**61**	**64**	**67**	**70**	**74**	**77**	**81**	**84**	**88**	**92**
40	**96**	**100**	**105**	106-108	108-112	110-116	114-119	118-123	122-127	125-131
50	130-135	134-139	138-143	142-147	147-151	151-155	156-160	160-164	165-168	170-172
60	175-177	180-181	**186**	187-190	189-195	191-199	193-204	195-208	199-213	204-217
70	208-222	212-227	217-231	222-236	227-241	232-246	237-251	242-255	247-260	252-265
80	257-270	262-275	268-280	273-285	279-291	284-296	290-301	296-306	302-311	308-316
90	315-322	318-327	322-332	325-337	329-343	334-348	339-354	344-359	350-365	356-370
100	362-376	368-381	375-387	381-392	388-398	394-404	401-409	407-415	414-421	420-426
110	427-432	434-438	441-444	448-450	**456**	457-462	459-468	461-473	463-479	465-485
120	467-491	469-497	475-504	482-510	489-516	496-522	504-528	511-534	519-541	526-547
130	534-553	541-559	549-566	556-572	564-578	571-585	579-592	586-598	594-604	601-611
140	609-617	616-624	624-630	632-637	640-644	648-650	**657**	658-663	660-670	662-677
150	664-683	666-690	668-697	670-704	672-710	679-718	687-724	695-732	703-738	711-745
160	720-752	728-759	737-766	745-773	754-780	762-787	771-794	779-801	788-808	796-815
170	805-823	813-830	822-837	830-844	839-851	847-859	856-866	864-873	873-880	882-888
180	891-895	900-902	**910**	911-917	913-925	915-932	917-939	919-947	921-954	923-962
190	925-970	927-977	929-985	937-993	946-1000	954-1008	963-1015	972-1023	981-1030	990-1038
200	1000-1046									

Table B.2: Summary of known values $ex(n;5)$, $ex_l(n;5)$ and $ex_u(n;5)$, for $n \leq 200$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**7**	**8**	**9**
10	**11**	**12**	**14**	**15**	**17**	**18**	**20**	**22**	**23**	**25**
20	**27**	**29**	**31**	**33**	**36**	**37**	**39**	**41**	**43**	**45**
30	**47**	**49**	**51**	**53**	55-58	58-61	59-63	61-65	63-68	65-70
40	67-73	69-75	71-77	73-80	75-82	78-85	81-87	83-90	86-92	88-95
50	91-98	93-100	95-103	97-106	100-108	102-111	104-114	106-116	108-119	110-122
60	112-124	115-127	118-130	121-133	124-136	127-138	130-141	134-144	136-147	138-150
70	140-153	143-156	147-158	148-161	150-164	152-167	154-170	156-173	158-176	160-179
80	162-182	164-185	166-188	168-191	171-194	174-197	176-201	178-204	180-207	182-210
90	184-213	186-216	188-219	190-222	193-226	196-229	198-232	201-235	204-238	206-242
100	209-245	212-248	214-251	217-255	220-258	223-261	225-265	228-268	231-271	234-275
110	237-278	240-281	243-285	246-288	249-291	252-295	255-298	258-301	261-305	265-308
120	268-312	271-315	274-319	278-322	281-326	284-329	287-333	291-336	294-339	297-343
130	301-347	305-350	308-354	311-357	314-361	317-364	320-368	323-371	327-375	330-379
140	333-382	337-386	340-389	344-393	348-397	352-400	355-404	359-408	363-411	367-415
150	371-419	375-422	380-426	382-430	384-434	387-437	391-441	395-445	399-449	403-452
160	408-456	409-460	411-464	413-467	415-471	417-475	419-479	421-483	423-487	425-490
170	427-494	429-498	431-502	433-506	435-510	437-513	439-517	441-521	443-525	445-529
180	447-533	449-537	451-541	453-545	455-549	458-553	460-557	462-561	464-565	467-568
190	469-572	471-576	473-580	475-584	478-588	482-592	485-596	489-601	492-605	496-609
200	499-613									

Table B.3: Summary of known values $ex(n;6)$, $ex_l(n;6)$ and $ex_u(n;6)$, for $n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**9**
10	**10**	**12**	**13**	**14**	**16**	**18**	**19**	**20**	**22**	**24**
20	**25**	**27**	**29**	**30**	**32**	**34**	**36**	**38**	**40**	**42**
30	**45**	**46**	**47**	**49**	**51**	**53**	**55**	56-59	58-61	60-63
40	62-65	64-67	65-69	67-71	69-73	71-76	73-78	75-80	77-82	79-84
50	81-87	84-89	86-91	88-93	90-96	93-98	96-100	98-103	100-105	102-107
60	105-110	108-112	110-115	112-117	114-119	117-122	120-124	122-127	125-129	128-132
70	130-134	133-137	136-139	138-142	141-144	144-147	147-149	150-152	153-154	156-157
80	**160**	161-162	162-165	164-167	166-170	168-173	170-175	172-178	174-181	176-183
90	178-186	180-189	181-191	183-194	185-197	187-199	189-202	191-205	193-208	196-210
100	200-213	201-216	202-219	204-221	206-224	209-227	212-230	215-233	218-236	221-238
110	224-241	227-244	230-247	233-250	236-253	239-256	242-258	245-261	248-264	252-267
120	256-270	258-273	261-276	264-279	267-282	270-285	273-288	276-291	278-294	281-297
130	284-300	287-303	290-306	293-309	296-312	299-315	302-318	306-321	309-324	312-327
140	316-330	319-333	322-336	325-339	328-342	332-345	335-348	338-351	342-355	345-358
150	348-361	352-364	356-367	360-370	363-373	367-376	371-380	374-383	378-386	382-389
160	385-392	389-396	393-399	396-402	400-405	404-408	408-412	412-415	416-418	420-421
170	**425**	426-428	427-431	429-434	431-438	433-441	435-444	436-447	438-451	440-454
180	442-457	444-461	446-464	448-467	450-471	452-474	453-477	455-481	457-484	459-487
190	461-491	463-494	465-497	467-501	469-504	471-507	473-511	475-514	477-518	479-521
200	481-524									

Table B.4: Summary of known values $ex(n;7)$, $ex_l(n;7)$ and $ex_u(n;7)$, for $n \leq 200$.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**9**
10	**10**	**11**	**12**	**14**	**15**	**16**	**18**	**19**	**21**	**22**
20	**23**	**25**	**27**	**28**	**29**	**31**	**33**	34-35	35-37	37-39
30	39-40	40-42	42-44	43-45	45-47	47-49	48-51	50-52	52-54	54-56
40	55-58	57-59	58-61	60-63	62-65	64-67	65-69	67-70	69-72	71-74
50	73-76	75-78	77-80	78-82	80-84	81-86	83-87	85-89	87-91	88-93
60	90-95	91-97	93-99	95-101	97-103	99-105	100-107	102-109	103-111	105-113
70	107-115	108-117	110-119	112-121	114-123	115-125	117-128	118-130	120-132	122-134
80	124-136	125-138	127-140	129-142	131-144	133-146	134-149	136-151	138-153	140-155
90	141-157	143-159	145-161	147-164	149-166	151-168	153-170	155-172	157-174	159-177
100	161-179	163-181	164-183	166-185	168-188	170-190	172-192	173-194	175-197	177-199
110	179-201	181-203	183-206	185-208	187-210	188-212	190-217	192-219	194-222	196-222
120	198-224	200-226	202-229	204-231	206-233	208-235	210-238	212-240	213-242	215-245
130	217-247	219-249	221-252	223-254	225-256	227-259	229-261	231-264	233-266	235-268
140	237-271	239-271	241-275	243-278	245-280	247-283	249-285	251-287	253-290	255-292
150	257-295	259-297	261-300	263-302	265-304	267-307	269-309	272-312	274-314	275-317
160	277-319	279-322	281-324	283-326	285-329	287-331	289-334	291-336	293-339	295-341
170	297-344	299-346	301-349	303-351	305-354	307-356	309-359	311-361	313-364	315-366
180	317-369	319-371	321-374	323-376	325-379	327-381	329-384	331-387	333-389	335-392
190	337-394	339-397	341-399	343-402	345-404	347-407	349-410	351-412	353-415	355-417
200	357-420									

Table B.5: Summary of known values $ex(n;8)$, $ex_l(n;8)$ and $ex_u(n;8)$, for $n \leq 200$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**10**	**11**	**12**	**13**	**15**	**16**	**17**	**18**	**20**	**21**
20	**23**	**24**	**25**	**27**	**28**	**30**	**31**	**32**	**34**	**36**
30	37-38	38-40	40-41	42-43	43-44	44-46	46-48	48-49	49-51	50-53
40	52-54	54-56	55-58	57-59	59-61	60-63	61-64	63-66	64-68	66-69
50	68-71	70-73	71-75	72-76	74-78	76-80	78-82	79-84	81-85	83-87
60	85-89	86-91	88-93	90-94	92-96	94-98	96-100	98-102	100-104	102-105
70	105-107	106-109	107-111	109-113	110-115	112-117	114-118	115-120	116-122	118-124
80	120-126	121-128	123-130	124-132	126-134	128-136	129-138	131-140	133-142	134-144
90	136-146	138-147	139-149	141-151	143-153	144-155	146-157	148-159	149-161	151-163
100	152-165	154-167	156-169	158-171	159-173	161-175	163-178	165-180	167-182	168-184
110	170-186	172-188	174-190	176-192	178-194	179-196	181-198	183-200	185-202	186-204
120	188-206	189-208	191-211	193-213	195-215	196-217	198-219	200-221	202-221	203-225
130	205-227	207-230	208-232	210-234	212-236	214-238	216-240	218-242	220-245	222-247
140	224-249	226-251	228-253	230-255	232-257	234-260	235-262	237-264	239-266	241-268
150	243-271	246-273	247-275	248-277	249-279	250-282	252-284	254-286	256-288	258-290
160	259-293	261-295	263-297	265-299	267-302	269-304	271-306	273-308	275-310	277-313
170	279-315	281-317	283-319	285-322	287-324	289-326	291-328	293-331	295-333	297-335
180	299-338	301-340	303-342	305-344	307-347	308-349	310-351	312-354	314-356	316-358
190	318-360	320-363	322-365	324-367	326-370	328-372	330-374	332-377	334-379	336-381
200	338-384									

Table B.6: Summary of known values $ex(n;9)$, $ex_l(n;9)$ and $ex_u(n;9)$, for $n \leq 200$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**9**	**11**	**12**	**13**	**14**	**15**	**17**	**18**	**19**	**20**
20	**22**	**23**	**24**	**26**	**27**	**28**	**30**	**31**	**33**	**34**
30	35-36	36-38	38-39	39-41	41-42	42-43	43-45	44-46	46-48	47-50
40	49-51	50-53	52-54	53-56	54-57	56-59	57-60	59-62	60-63	62-65
50	63-67	65-68	66-70	67-71	69-73	71-75	72-76	73-78	75-79	76-81
60	78-83	79-84	81-86	82-88	84-89	86-91	87-93	88-94	90-96	91-97
70	93-99	94-101	96-103	97-104	99-106	100-108	102-109	103-111	105-113	107-114
80	108-116	110-118	111-120	113-121	114-123	116-125	117-127	119-128	121-130	123-132
90	125-133	127-135	129-137	130-139	132-141	134-142	136-144	138-146	140-148	142-149
100	144-151	146-153	147-155	149-157	151-158	153-160	155-162	157-164	159-166	161-167
110	163-169	165-171	168-173	169-175	171-176	172-178	174-180	175-182	177-184	178-186
120	180-187	181-189	183-191	184-193	186-195	188-197	189-198	190-200	192-202	193-204
130	195-206	196-208	198-210	199-212	201-213	202-315	204-317	205-219	207-221	208-223
140	210-225	211-227	213-229	214-230	216-232	217-234	219-236	220-238	222-240	223-242
150	225-244	226-246	228-248	229-249	231-251	232-253	234-255	236-257	238-259	239-261
160	240-263	242-265	244-267	245-269	247-271	248-273	250-275	251-277	253-279	255-280
170	256-282	258-284	259-286	261-288	262-290	264-292	266-294	267-296	269-298	270-300
180	272-302	273-304	275-306	276-308	278-310	279-312	281-314	282-316	284-318	285-320
190	287-322	289-324	291-326	292-328	294-330	295-332	297-334	298-336	300-338	302-340
200	303-342									

Table B.7: Summary of known values $ex(n;10)$, $ex_l(n;10)$ and $ex_u(n;10)$, for $n \leq 200$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
10	**9**	**10**	**12**	**13**	**14**	**15**	**16**	**18**	**19**	**20**
20	**21**	**22**	**24**	**25**	**27**	**28**	**29**	**30**	**32**	**33**
30	**34**	**36**	**37**	38-39	40-41	**42**	43-44	44-45	45-46	46-48
40	48-49	49-51	50-52	52-54	53-55	54-56	56-58	57-60	58-61	60-63
50	61-64	62-66	64-67	66-69	67-70	68-72	70-73	71-75	72-76	74-78
60	75-79	80-81	81-82	82-84	84-85	85-87	87-89	88-90	89-92	91-93
70	92-95	94-96	95-98	97-100	98-101	100-103	101-104	103-106	104-108	106-109
80	107-111	109-113	110-114	112-116	113-117	115-119	116-121	118-122	119-124	121-126
90	122-127	124-129	125-131	127-132	128-134	130-136	132-137	133-139	135-141	136-142
100	138-144	139-146	141-147	142-149	144-151	146-152	148-154	150-156	152-157	154-159
110	156-161	158-163	160-164	162-166	164-168	166-169	168-171	170-173	172-175	174-176
120	176-178	178-180	180-182	182-183	184-185	186-187	**189**	**190**	191-192	192-194
130	193-196	195-197	196-199	198-201	199-203	200-204	202-206	203-208	205-210	206-211
140	207-213	209-215	210-217	212-218	213-220	214-222	216-224	217-226	219-227	220-229
150	221-231	223-233	224-235	226-236	227-238	229-240	230-242	232-244	234-245	236-247
160	238-249	240-251	243-253	244-254	245-256	247-258	248-260	249-262	252-264	253-265
170	254-267	256-269	257-271	258-273	259-275	260-276	261-278	262-280	264-282	266-284
180	268-286	270-288	273-289	274-291	276-293	277-295	278-297	279-299	280-301	281-302
190	283-304	285-306	288-308	289-310	290-312	292-314	293-315	294-317	295-319	296-321
200	298-323									

Table B.8: Summary of known values $ex(n;11)$, $ex_l(n;11)$ and $ex_u(n;11)$, for $n \leq 200$.